

A bound on the mutual information, and properties of entropy reduction, for quantum channels with inefficient measurements

Kurt Jacobs

Centre for Quantum Computer Technology, Centre for Quantum Dynamics,
School of Science, Griffith University, Nathan 4111, Brisbane, Australia

(Received 7 December 2004; accepted 1 November 2005; published online 13 January 2006)

The Holevo bound is a bound on the mutual information for a given quantum encoding. In 1996 Schumacher, Westmoreland, and Wootters [Phys. Rev. Lett. **76**, 3452 (1996)] derived a bound that reduces to the Holevo bound for complete measurements, but that is tighter for incomplete measurements. The most general quantum operations may be both incomplete and inefficient. Here we show that the bound derived by SWW can be further extended to obtain one that is yet again tighter for inefficient measurements. This allows us, in addition, to obtain a generalization of a bound derived by Hall, and to show that the average reduction in the von Neumann entropy during a quantum operation is concave in the initial state, for all quantum operations. This is a quantum version of the concavity of the mutual information. We also show that both this average entropy reduction and the mutual information for pure state ensembles, are Schur concave for unitarily covariant measurements; that is, for these measurements, information gain increases with initial uncertainty. © 2006 American Institute of Physics.
[DOI: [10.1063/1.2158433](https://doi.org/10.1063/1.2158433)]

I. INTRODUCTION

The celebrated Holevo bound, conjectured by Gordon¹ and Levitin² and proved by Holevo in 1973³ gives a bound on the information that may be transmitted from A to B (strictly, the *mutual information*, M , between A and B) when A encodes information in a quantum system using a set of states $\{\rho_i\}$, chosen with probabilities $\{P(i)\}$, and B makes a subsequent measurement upon the system. The Holevo bound is

$$M(I:J) \leq \chi \equiv S(\rho) - \sum_i P(i)S(\rho_i), \quad (1)$$

where $\rho = \sum_i P(i)\rho_i$ (and which we will refer to as the *ensemble state*). We write the mutual information as $M(I:J)$ to signify that it is the mutual information between the random variables I and J , whose values i and j label, respectively, the encoding used by A, and the outcome of the measurement made by B. More recent proofs of the Holevo bound may be found in Refs. 4–6. The bound is achieved if and only if the encoding states, ρ_i , commute with each other, and the receiver, B, makes a von Neumann measurement in the basis in which they are diagonal. (A von Neumann measurement is one that projects the system onto one of a complete set of mutually orthogonal states. In this case the set of states is chosen to be the basis in which the coding states are diagonal.) With this choice of coding states and measurement the channel is classical, in that it can be implemented with a classical system. The Holevo bound takes into account that the sender may only be able to send mixed states, and this mixing reduces the amount of information that can be transmitted. However, if the receiver is not able to perform measurements that always project the system to a pure state (so-called *complete* measurements), then, in general, the information will be further reduced. In 1996 Schumacher, Westmoreland, and Wootters showed that when the receiver

er's measurement is incomplete, it is possible to take this into account and derive a more stringent bound on the information. If the receiver's measurement is the POVM described by the operators $\{A_j\}$ (with $\sum_j A_j^\dagger A_j = 1$), so that the measurement outcomes are labeled by the index j , then the SWW bound is⁶

$$M(I:J) \leq \chi - \sum_j P(j) \chi_j, \quad (2)$$

where $P(j)$ is the probability of outcome j ,⁷ and χ_j is the Holevo quantity for the ensemble that the system remains in (from the point of view of the receiver), given outcome j . This bound can be at least partially understood by noting that if the system still remains in some ensemble of possible states after the measurement, then future measurements can potentially extract further information about the encoding, and so the information obtained by the first measurement must therefore be less than the maximum possible at least by this amount. What the SWW bound tells us is that the bound on the information is reduced not only by the amount of information that could be further extracted after outcome j has been obtained, but by the Holevo bound on this information, χ_j .

If the initial state of the system is ρ_i , then after outcome j the final state of the system is given by $\tilde{\rho}_{j|i} = A_j \rho_i A_j^\dagger / \text{Tr}[A_j^\dagger A_j \rho_i]$. Thus the states that make up the final ensemble that remains after outcome j , are $\{\tilde{\rho}_{j|i}\}$, and the probability of each state in the ensemble is $P(i|j) = P(j|i)P(i)/P(j)$, with $P(j|i) = \text{Tr}[A_j^\dagger A_j \rho_i]$. The Holevo quantity for ensemble j is thus

$$\chi_j = S(\tilde{\rho}_j) - \sum_i P(i|j) S(\tilde{\rho}_{j|i}), \quad (3)$$

where $\tilde{\rho}_j = A_j \rho A_j^\dagger / \text{Tr}[A_j^\dagger A_j \rho]$. If at least one of the measurement operators A_j are higher than rank 1, then the measurement is incomplete. If the measurement is complete, then for each j all the final states $\tilde{\rho}_{j|i}$ are identical, χ_j is zero and the SWW bound reduces to the Holevo bound.

The most general kind of measurement can also be inefficient. A measurement is described as inefficient if the observer does not have full information regarding which of the outcomes actually occurred. The name *inefficient* comes from that fact that the need to consider such measurements first arose in the study of inefficient photo-detectors.⁸ An inefficient measurement may be described by labeling the measurement operators with two indices, so that we have A_{kj} . The receiver has complete information about one of the indices, j , but no information about the other, k .⁹ As a result, the final state for each j (given the value of i) is now

$$\rho'_{j|i} = \sum_k P(k|j) \frac{A_{kj} \rho_i A_{kj}^\dagger}{\text{Tr}[A_{kj}^\dagger A_{kj} \rho_i]}. \quad (4)$$

Since inefficiency represents a loss of information, we wish to ask whether it is possible to take this into account and obtain a more stringent bound on the mutual information. If we merely apply the SWW bound to the measurement A_{kj} , then the bound involves the Holevo quantities of the ensembles that remain when both the values of k and j are known (the final ensembles that result from the efficient measurement). That is

$$M(I:J) \leq \chi - \sum_{kj} P(k,j) \chi_{kj}. \quad (5)$$

One therefore wishes to know whether it is possible to derive a bound that instead involves the Holevo quantities of the ensembles that remain after the inefficient measurement is made, that is, for the receiver who only has access to j .

In the first part of this paper we answer this question in the affirmative—for an inefficient measurement where the known outcomes are labeled by j , the bound given by Eq. (2) *remains true*, where now the χ_j are the Holevo quantities for the ensemble of states $\rho'_{j|i}$ that result from the inefficient measurement.

In the second part of the paper, we consider the average reduction in the von Neumann entropy induced by a measurement:

$$\langle \Delta S(\rho) \rangle \equiv S(\rho) - \sum_i P(j) S(\rho'_j). \quad (6)$$

Here ρ'_j is the state that results from outcome j , given that the initial state is ρ . Since the von Neumann entropy is a measure of how much we know about the state of the system, this is the difference between what we knew about the system state before we made the measurement, and what we know (on average) about the system state at the end of the measurement; it thus measures how much we learn about the final state of the system. Equivalently, it can be said to measure the degree of “state-reduction” that the measurement induces.

While it is the mutual information that is important for communication, the reduction in the von Neumann entropy is important for feedback control. Feedback control is the process of performing a sequence of measurements on a system, and applying unitary operations after each measurement in order control the evolution of the system. Such a procedure is useful for controlling systems that are driven by noise. If the ability to perform unitary operations is unlimited, then the von Neumann entropy provides a measure of the level of control that can be achieved: if the system has maximal entropy then the unitary operations have no effect on the system state whatsoever; conversely, if the state is pure then the system can be controlled precisely - that is, any pure state can be prepared. Thus the entropy measures the extent to which a pure state, or pure evolution can be obtained, and thus the level of predictability which can be achieved over the future behavior of the system.¹⁰ The primary role of measurement in feedback control is therefore to reduce the entropy of the system. As such, the average reduction in von Neumann entropy provides a ranking of the effectiveness of different measurements for feedback control, other things being equal. Further details regarding quantum feedback control and von Neumann entropy can be found in Ref. 11.

The entropy reduction is also relevant to the transformation of pure-state entanglement, since the von Neumann entropy measures the entanglement of pure states. As a result this quantity gives the amount by which pure-state entanglement is broken by a local measurement.

We give two corollaries of the general information bound derived in the first part that involve $\langle \Delta S(\rho) \rangle$. The first is a generalization of a bound derived by Hall^{12,13} to inefficient measurements. Hall’s bound states that for efficient measurements the mutual information is bounded by $\langle \Delta S(\rho) \rangle$. We show that for inefficient measurements this becomes

$$M(I:J) \leq \langle \Delta S(\rho) \rangle - \sum_i P(i) \langle \Delta S(\rho_i) \rangle, \quad (7)$$

where $\langle \Delta S(\rho_i) \rangle$ is the average entropy reduction that would have resulted if the initial state had been ρ_i , and as above $\rho = \sum_i P(i) \rho_i$.

The second is the fundamental property that, for all quantum operations, the average reduction in von Neumann entropy is concave in the initial state ρ . That is,

$$\langle \Delta S(\rho) \rangle \geq \sum_i P(i) \langle \Delta S(\rho_i) \rangle. \quad (8)$$

Finally, in the third part of this paper, we use the above result to show that for measurements that are uniform in their sensitivity across state space (that is, measurements that are unitarily covariant), the amount that one learns about the final state always increases with the initial uncertainty, where this uncertainty is characterized by *majorization*. This is a quantum version of the much simpler classical result (which we also show) that the mutual information always increases with the initial uncertainty for classical measurements that are permutation symmetric. In addition, we show that, for unitarily covariant measurements, the mutual information for pure-state ensembles also has this property. One can sum up these results by saying that the statement

that information gain increases with initial uncertainty can fail to hold only if the measurement is asymmetric in its sensitivity.

II. AN INFORMATION BOUND FOR GENERAL QUANTUM OPERATIONS

We now show that the bound proved by SWW can be generalized to obtain a more stringent bound for channels in which the receiver's measurement is inefficient. To show this, it turns out that we can use the same method employed by SWW, but with the addition of an extra quantum system, which allows us to include the inefficiency of the measurement.

Theorem 1: *For a quantum channel in which the encoding ensemble is $\epsilon = \{P(i), \rho_i\}$, and the measurement performed by the receiver is described by operators A_{kj} ($\sum_k A_{kj}^\dagger A_{kj} = 1$), where the measurement is, in general, inefficient so that the receiver knows j but not k ; then the mutual information, $M(I:J)$, is bounded such that*

$$M(I:J) \leq \chi - \sum_j P(j) \chi_j, \tag{9}$$

where $P(j)$ is the overall probability for outcome j , $\chi = S(\rho) - \sum_i P(i) S(\rho_i)$ is the Holevo quantity for the initial ensemble, and

$$\chi_j = S(\sigma_j) - \sum P(i|j) S(\sigma_{ji}), \tag{10}$$

is the Holevo quantity for the ensemble, ϵ_j , that remains (from the point of view of the receiver) once the measurement has been made, so that the receiver has learned the outcome j , but not the value of k . Here the receiver's overall final state is

$$\sigma_j = \frac{\sum_k A_{kj} \rho A_{kj}^\dagger}{P(j)} = \sum_{ik} P(i, k|j) \sigma_{kji}, \tag{11}$$

where $P(i, k|j)$ is the probability for both i and outcome k given j , and σ_{kji} is the final state that results given the initial state ρ_i , and both outcomes j and k . The remaining ensemble $\epsilon_j = \{P(i|j), \sigma_{ji}\}$, where

$$\sigma_{ji} = \sum_k P(k|j, i) \sigma_{kji} = \frac{\sum_k A_{kj} \rho_i A_{kj}^\dagger}{P(j|i)}, \tag{12}$$

and where $P(k|j, i)$ is the probability for outcome k given j and the initial state ρ_i .

Proof: We begin by collecting various key facts. The first is that any efficient measurement on a system Q , described by $N = N_1 N_2$ operators, A_{kj} ($j = 1, \dots, N_1$ and $k = 1, \dots, N_2$) can be obtained by bringing up an auxiliary system A of dimension N , performing a unitary operation involving Q and A , and then making a von Neumann measurement on A .^{14,15} If the initial state of Q is $\rho^{(Q)}$, then the final joint state of A and Q after the von Neumann measurement is

$$\sigma^{(AQ)} = |kj\rangle\langle kj|^{(A)} \otimes \frac{A_{kj} \rho^{(Q)} A_{kj}^\dagger}{P(k, j)}. \tag{13}$$

where $|kj\rangle$ is the state of A selected by the von Neumann measurement. The second fact is that the state that results from discarding all information about the measurement outcomes k and j can be obtained by performing a unitary operation between A and another system E that perfectly correlates the states $|kj\rangle$ of A with orthogonal states of E , and then tracing out E . The final key fact we require is a result proven by SWW,⁶ which is that the Holevo χ quantity is nonincreasing under partial trace. That is, if we have two quantum systems A and B , and an ensemble of states $\rho_i^{(AB)}$ with associated probabilities P_i , then

$$\chi^{(A)} = S(\rho^{(A)}) - \sum_i S(\rho_i^{(A)}) \leq S(\rho^{(AB)}) - \sum_i S(\rho_i^{(AB)}) = \chi^{(AB)}, \tag{14}$$

where $\rho_i^{(A)} = \text{Tr}_B[\rho_i^{(AB)}]$. To prove this result SWW use strong subadditivity.¹⁶

We now encode information in system Q using the ensemble ϵ , and consider the joint system that consists of the three systems Q, A, E and a fourth system M , with dimension N_1 . We now start with A, E and M in pure states, so that the Holevo quantity for the joint system is $\chi^{(Q A E M)} = \chi^{(Q)}$. We then perform the required unitary operation between Q and A , and a unitary operation between A and E that perfectly correlates the states $|k_j\rangle^{(A)}$ of A with orthogonal states of E . Unitary operations do not change the Holevo quantity. Then we trace over E , so that we are left with the state

$$|\psi\rangle\langle\psi|^{(M)} \otimes \sum_{jk} P(k,j) |k,j\rangle\langle k,j|^{(A)} \otimes \frac{A_{kj} \rho^{(Q)} A_{kj}^\dagger}{P(k,j)}. \tag{15}$$

After the two unitaries and the partial trace over E , the Holevo quantity for the remaining systems, which we will denote by $\chi^{(Q A M)}$, satisfies $\chi^{(Q A M)} \leq \chi^{(Q A E M)} = \chi^{(Q)}$. We now perform one more unitary operation, this time between M and A , so that we correlate the states of M , which we denote by $|j\rangle\langle j|^{(M)}$ with the second index of the states of A , giving

$$\sum_j |j\rangle\langle j|^{(M)} \otimes \sum_k P(k,j) |k,j\rangle\langle k,j|^{(A)} \otimes \sigma_{kj}^{(Q)} \tag{16}$$

where $\sigma_{kj}^{(Q)} = A_{kj} \rho^{(Q)} A_{kj}^\dagger / P(k,j)$ is the final state resulting from knowing both outcomes k and j , with no knowledge of the initial choice of i . Finally we trace out A , leaving us with the state

$$\sigma^{(QM)} = \sum_j |j\rangle\langle j|^{(M)} \otimes \sum_k P(k,j) \sigma_{kj}^{(Q)}. \tag{17}$$

After this final unitary, and the partial trace over A , the Holevo quantity for the remaining systems Q and M , which we will denote by $\chi^{(QM)}$, satisfies $\chi^{(QM)} \leq \chi^{(Q A M)} \leq \chi^{(Q)}$. We have gone through the above process using the initial state ρ , but we could just as easily have started with any of the initial states, ρ_i , in the ensemble, and we will denote the final states that we obtain using the initial state ρ_i as $\sigma_i^{(QM)}$. Calculating $\chi^{(QM)}$, we have

$$\chi^{(QM)} = S(\sigma^{(QM)}) - \sum_i P(i) S(\sigma_i^{(QM)}) = H[J] - \sum_i P(i) H[J|i] + \sum_j P(j) \left[S(\sigma_j) - \sum_i P(i|j) S(\sigma_{ji}) \right] \tag{18}$$

$$= M(J:I) + \sum_j P(j) \chi_j^{(Q)} \leq \chi^{(Q)}. \tag{19}$$

Rearranging this expression gives the desired result. □

III. PROPERTIES OF ENTROPY REDUCTION

We now rewrite the above information bound using the fact that $P(i|j)P(j) = P(j|i)P(i)$. The result is

$$M(I:J) \leq \langle \Delta S(\rho) \rangle - \sum_i P(i) \langle \Delta S(\rho_i) \rangle, \tag{20}$$

where $\rho = \sum_i P_i \rho_i$. Ozawa has shown that for efficient measurements $\langle \Delta S(\rho) \rangle$ is always positive¹⁷ (for more recent proofs of this result see Refs. 18 and 19). For efficient measurements, Eq. (20) is therefore in general stronger than, and gives immediately, Hall's bound,^{12,13} which states that the mutual information is bounded by the reduction in the von Neumann entropy. The inequality in

Eq. (20) is then a generalization of Hall's bound to inefficient measurements. Since the mutual information is always positive, but for inefficient measurements the reduction in the von Neumann entropy can be negative (that is, the entropy of the quantum state can *increase* as a result of the measurement), the relation

$$M(I:J) \leq \langle \Delta S(\rho) \rangle \quad (21)$$

is not necessarily satisfied for such measurements. However, Eq. (20) tells us that if the entropy of the initial state, ρ , does increase, the average increase in the entropy for each of the coding states ρ_i is always *more* than this by at least the mutual information.

The second result that we obtain from Eq. (20) is that, because the mutual information is non-negative, we have

$$\langle \Delta S(\rho) \rangle \geq \sum_i P(i) \langle \Delta S(\rho_i) \rangle. \quad (22)$$

That is, the reduction in the von Neumann entropy is concave in the initial state. This parallels the fact that the mutual information is also concave in the initial state.

The fact that this is true for inefficient measurements means that once we have made an efficient measurement, no matter what information we throw away regarding the final outcomes (i.e., which outcomes we average over), $\langle \Delta S(\rho) \rangle$ is always greater than the average of the entropy reductions that would have been obtained through measurement in each of the coding states, when we throw away the same information regarding the measurement results.

IV. INFORMATION GATHERING AND STATE-SPACE SYMMETRY

In this section we show that measurements whose ability to extract information is uniform over the available state space (that is, does not vary from point to point in the state-space) always extract more information (strictly, never extract less information) the less that is known before the measurement is made. Thus, in this sense, one may regard "the more you know, the less you get" as a fundamental property of measurement. We will show that this is true both for the information obtained regarding the final state [being $\langle \Delta S(\rho) \rangle$], and the mutual information for a measurement on an ensemble of pure states. Here we will consider efficient measurements only; no doubt inefficient measurements will also have this property, but only if the information that is thrown away is also uniform with respect to the state space, and we do not wish to burden the treatment with this additional complication.

To proceed, we must make precise the notion that the sensitivity of a measurement is uniform over state space. This is captured by stating that such a measurement should be invariant under reversible transformations of the state space. For classical measurements (which are simply quantum measurements in which all operators and density matrices commute²⁰) this means that the set of measurement operators is invariant under all permutations of the classical states: we will refer to these as *completely symmetric* measurements. Note that in this classical case, this is equivalent to saying that the measurement distinguishes all states from all other states equally well. The quantum generalization of this is invariance under all unitary transformations. Such measurements are referred to as being unitarily covariant.^{21,22}

We must also quantify what we mean by the observer's lack of knowledge, or uncertainty, before the measurement is made. This is captured by the simple and elegant concept of *majorization*.^{23,24} If two sets of probabilities $p \equiv \{P_i\}$ and $q \equiv \{Q_i\}$ satisfy the set of relations

$$\sum_{i=1}^k P_i \geq \sum_{i=1}^k Q_i, \quad \forall k, \quad (23)$$

where it is understood that the elements of both sets have been placed in decreasing order (e.g., $P_i > P_{i+1}$, $\forall i$), then p is said to majorize q , and this is written $q < p$. While at first Eq. (23) looks a little complicated, a few moments consideration reveals that it captures precisely what one

means by uncertainty—if p majorizes q , then p is more sharply peaked than q , and consequently describes a state of knowledge containing less uncertainty. What is more, majorization implies an ordering with Shannon entropy $H[\cdot]$. That is, if p majorizes q , then $H[p] \leq H[q]$.^{23,24}

In a sense, majorization is a more basic notion of uncertainty than entropy in that it captures that concept alone—the Shannon entropy, on the other hand, characterizes the more specific notion of information. To characterize the uncertainty of a density matrix, we can apply majorization to the vector consisting of its eigenvalues. If ρ and σ are density matrices, then we will write $\sigma \prec \rho$ if ρ 's eigenvalues majorize σ 's. Various applications have been found for majorization in quantum information theory.^{18,19,25–28}

We thus desire to show that for measurements with the specified symmetry, $\langle \Delta S(\sigma) \rangle \geq \langle \Delta S(\rho) \rangle$ whenever $\sigma \prec \rho$ (and similarly for the mutual information). Functions with this property [of which the von Neumann entropy, $S(\rho)$, is one example] are referred to as being *Schur concave*. To show that a function is Schur concave, it is sufficient to show that it is concave, and symmetric in its arguments,^{23,24} which in our case are the eigenvalues of the density matrix ρ (if our functions did not depend only on the eigenvalues of ρ , then they could not be Schur concave, since the majorization condition only involves these eigenvalues).

The desired result for classical completely symmetric measurements is now immediate. In the classical case the mutual information is the unique measure of information gain, and $M(I:J) = \langle \Delta S(\rho) \rangle$. The mutual information is concave in the initial classical probability vector $\mathbf{P} = (P_1, \dots, P_n)$ (being the vector of the eigenvalues of ρ in our quantum formalism), as is indeed implied by the concavity of $\langle \Delta S(\rho) \rangle$. Since all operators commute with the density matrix, $\langle \Delta S(\rho) \rangle$ is only a function of the $\{P_{ij}\}$. From the form of $\langle \Delta S(\rho) \rangle$ we see that a permutation of the elements of \mathbf{P} is equivalent to a permutation applied to the measurement operators, and since these are invariant under such an operation, $\langle \Delta S(\rho) \rangle$, and thus $M(I:J)$, is a symmetric function of its arguments. Thus $M(I:J)$ is Schur concave.

The Schur concavity of $\langle \Delta S(\rho) \rangle$ for unitarily covariant (UC) quantum measurements is just as immediate. Because of the unitary covariance of the measurement, we see from the form of $\langle \Delta S(\rho) \rangle$ that it is invariant under a unitary transformation of ρ . As a result, it only depends upon the eigenvalues of ρ . Since the permutations are a subgroup of the unitaries, it is also a symmetric function of its arguments (the eigenvalues), and thus Schur concave.

We wish finally to show that the mutual information is also Schur concave in ρ for unitarily covariant measurements on ensembles of pure states. This requires a little more work. First, we need to show that once we have fixed a set of encoding states, the mutual information is concave in the vector of the ensemble probabilities $P(i)$. This is straightforward if we first note that the mutual information, because it is, in fact, symmetric between i and j , can be written in the reverse form

$$M(I:J) = H[P(j)] - \sum_j P(i)H[P(j|i)]. \quad (24)$$

Since, for a fixed measurement, the mutual information is a function of the ensemble probabilities, we will write it as $M(\{P(i)\})$. Denoting the pure states in the encoding ensemble as $\rho_i = |\psi_i\rangle\langle\psi_i|$, and choosing the ensemble state $\rho = \sum_k P_k \sigma_k$, where the σ_k are built from the encoding states so that $\sigma_k = \sum_i P_{i|k} |\psi_i\rangle\langle\psi_i|$, then

$$\begin{aligned} M(\{P(i)\}) &= H\left[\sum_k P(k)P(j|k)\right] - \sum_i \sum_k P(i|k)P(k)H[P(j|i)] \\ &\geq \sum_k P(k)H[P(j|k)] - \sum_k P(k) \sum_i P(i|k)H[P(j|i)] \\ &= \sum_k P(k)M(\{P(i|k)\}), \end{aligned} \quad (25)$$

being the desired concavity relation. The inequality in the third line is merely a result of the

concavity of the Shannon entropy. Note that while we have written the measurement's outcomes explicitly as being discrete in the above derivation, the result also follows if they are a continuum (as in the case of UC measurements) by replacing the relevant sums with integrals.

Now we need to note some further points about UC measurements: A UC measurement may be generated by taking all unitary transformations of any single operator A , and dividing them by a common normalization factor. The resulting measurement operators are thus $A_U \propto UAU^\dagger$, where U ranges over all unitaries. The normalization for the A_U comes from $\int UA^\dagger AU^\dagger d\mu(U) = \text{Tr}[A^\dagger A]I$, where $d\mu(U)$ is the (unitarily invariant) Haar measure^{22,29} over unitaries.

It is not hard to show that all UC measurements can be obtained by *mixing* different UC measurements, each generated by a different operator. (Mixing a set of measurements means assigning to each a probability, and then making one measurement from the set at random based on these probabilities³⁰).

Next, we need to show that for all UC measurements the mutual information depends only on the eigenvalues of the ensemble density matrix, and we state this as the following lemma.

Lemma 1: The mutual information for a UC measurement on a pure-state ensemble, $\epsilon = \{P(i), |\psi_i\rangle\}$ depends on the ensemble only through the eigenvalues of the density matrix $\rho = \sum_i P(i) |\psi_i\rangle\langle\psi_i|$.

Proof: We first show this for UC measurements generated from a single operator. Writing the mutual information in the reverse form, one has

$$M(I:J) = H[P(U)] - \sum_i P_i H[P(U|i)], \quad (26)$$

where U is the continuum index for the measurement operators (and thus the measurement outcomes) that are $A_U = UAU^\dagger$ for some appropriately normalized A . Naturally all this means is that $P(U|i)$ is a function of U , where U ranges over all unitaries. Since the measurement is unitarily covariant, $H[P(U|i)]$ is the same for all initial states $|\psi_i\rangle$, and therefore the second term is the same for all initial ensembles. Thus M depends only on the first term $H[P(U)] = H[\text{Tr}[UA^\dagger AU^\dagger \rho]]$, which depends only on ρ , and is invariant under all unitary transformations of ρ . Thus M depends only on the eigenvalues of ρ . Since the mutual information for a mixture of measurements is merely a function of the respective mutual informations for each measurement (in particular, it is a linear combination of them), the result holds for all UC measurements. \square

Since M depends only on ρ , in establishing the Schur concavity of M with respect to ρ , we need only consider one ensemble for each ρ . We therefore choose the eigenensemble $\{\lambda_i, |\phi_i\rangle\}$, where λ_i and $|\phi_i\rangle$ are the eigenvalues and eigenvectors of ρ , respectively. We know that the mutual information is concave in the vector of initial ensemble probabilities, and for the ensemble we have chosen, the initial probabilities are the eigenvalues of ρ . As a result, the mutual information is concave in the eigenvalues of ρ . Since M is invariant under unitary transformations, and since unitary transformations include permutations as a subgroup, it is also a symmetric function of the eigenvalues. Thus M is Schur concave.

V. CONCLUSION

In using a quantum channel, if there are limitations on the completeness (or alternatively the *strength*, in the terminology of Ref. 19) or efficiency of the measurements that the receiver can perform, then it is possible to give a bound on the mutual information that is stronger than the Holevo bound. Further, this bound has a very simple form in terms of the Holevo χ quantity, and the χ quantities of the ensembles, one of which remains after the measurement is made.

This bound also allows us to obtain a relationship between the mutual information and the average von Neumann entropy reduction induced by a measurement, and encompasses the fact that this von Neumann entropy reduction is concave in the initial state.

From the concavity of the mutual information and the von Neumann entropy reduction, it follows that these quantities are Schur concave (the former naturally for pure-state ensembles) for completely symmetric classical measurements, and for unitarily covariant quantum measurements.

Thus, the possibility that either of these kinds of information gain *decreases* with increasing initial uncertainty is associated with the asymmetry of the measurement in question.

ACKNOWLEDGMENTS

The author would like to thank Gerard Jungman, Howard Barnum, Howard Wiseman, Terry Rudolph, and Michael Hall for helpful discussions. The author is also grateful both to Vlatko Vedral for hospitality during a visit to Imperial College, and Lucien Hardy for hospitality during a visit to the Perimeter Institute, where some of the initial stages of this work was carried out. This work was supported by the Australian Research Council and the State of Queensland. *Note added:* After submitting this manuscript, which was first posted as eprint quant-ph/0412006, it was brought to my attention that the work presented here overlaps with concurrent work by Barchielli and Lupeiri (quant-ph/0409019 and quant-ph/0412116).

- ¹J. P. Gordon in *Quantum Electronics and Coherent Light*, Proceedings of the International School of Physics "Enrico Fermi" XXXI, edited by P. A. Miles (Academic, New York, 1964).
- ²L. B. Levitin, in *Proceedings of the All-Union Conference on Information Complexity and Control in Quantum Physics*, (Mockva-Tashkent, Tashkent, 1969), Sec. II (in Russian); L. B. Levitin, in *Information Complexity and Control in Quantum Physics*, edited by A. Blaquiere, S. Diner, and G. Lochak (Springer-Verlag, New York, 1987), pp. 15–47.
- ³A. S. Holevo, *Probl. Peredachi Inf.* **9**, 3 (1973) A. S. Holevo, [*Probl. Inf. Transm. (USSR)* **9**, 177 (1973)].
- ⁴H. P. Yuen and M. Ozawa, *Phys. Rev. Lett.* **70**, 363 (1993).
- ⁵C. A. Fuchs and C. M. Caves, *Phys. Rev. Lett.* **73**, 3047 (1994).
- ⁶B. Schumacher, M. Westmoreland, and W. K. Wootters, *Phys. Rev. Lett.* **76**, 3452 (1996).
- ⁷While it is an abuse of notation to denote the ensemble probabilities by $P(i)$, and the (in general, unrelated) outcome probabilities by $P(j)$, we use it systematically throughout, since we feel it keeps the notation simpler, and thus ultimately clearer.
- ⁸H. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer-Verlag, Berlin, 1993); H. M. Wiseman and G. J. Milburn, *Phys. Rev. A* **47**, 642 (1993).
- ⁹If the observer has only partial information about the outcome of a measurement, then if we label the outcomes by n (with associated measurement operators B_n), the most general situation is one in which the observer knows instead the value of a second variable m , where m is related to n by an arbitrary conditional probability $P(m|n)$. This general case is encompassed by the two-index formulation we use in the text. To see this, one sets $k=n$, $j=m$ and chooses $A_{nm}(\equiv A_{kj})=\alpha_{nm}B_n$. Then by giving the observer complete knowledge of j , and no knowledge of k , we reproduce precisely the general case described above by choosing α_{nm} so that $|\alpha_{nm}|^2=P(m|n)$.
- ¹⁰The von Neumann entropy is not the only quantity that can be used to measure the achieved level of control. The von Neumann entropy specifically gives the minimum possible entropy of the results of a measurement on the system. It therefore measures the maximum information (strictly, the minimum information deficit) that the user who is performing the control has about of the future behavior of the system under measurement. An example of another measure of control is the maximum eigenvalue of the density matrix. Under the assumption that all unitary operations are available to the controller, this measures the probability that the controlled system will be found in the desired state.
- ¹¹A. Doherty, K. Jacobs, and G. Jungman, *Phys. Rev. A* **63**, 062306 (2001).
- ¹²M. J. W. Hall, *Phys. Rev. A* **55**, 100 (1997).
- ¹³K. Jacobs, *Phys. Rev. A* **68**, 054302(BR) (2003).
- ¹⁴B. Schumacher, *Phys. Rev. A* **54**, 2614 (1996).
- ¹⁵K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory*, Lecture Notes in Physics Vol. 190 (Springer-Verlag, Berlin, 1983).
- ¹⁶E. H. Lieb, *Adv. Math.* **11**, 267 (1973); E. H. Lieb and M. B. Ruskai, *Phys. Rev. Lett.* **30**, 434 (1973); E. H. Lieb and M. B. Ruskai, *J. Math. Phys.* **14**, 1938 (1973); In addition, a much simpler proof of strong subadditivity has been obtained by Petz [*Rep. Math. Phys.* **23**, 57 (1986)] and is described in M. A. Nielsen and D. Petz, e-print: quant-ph/0408130.
- ¹⁷M. Ozawa, *J. Math. Phys.* **27**, 759 (1986).
- ¹⁸M. A. Nielsen, *Phys. Rev. A* **63**, 022114 (2001).
- ¹⁹C. A. Fuchs and K. Jacobs, *Phys. Rev. A* **63**, 062305 (2001).
- ²⁰A discussion of this point may be found in K. Jacobs, *Quantum Inf. Process.* **1**, 73 (2002).
- ²¹H. Barnum, "Information-disturbance tradeoff in quantum measurement on the uniform ensemble and on the mutually unbiased bases," e-print: quant-ph/0205155.
- ²²G. Cassinelli, E. De Vito, A. Toigo, "Positive operator valued measures covariant with respect to an irreducible representation," e-print: quant-ph/0302187.
- ²³A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications* (Academic, New York, 1979).
- ²⁴R. Bhatia, *Matrix Inequalities* (Springer-Verlag, Berlin, 1997).
- ²⁵M. A. Nielsen, *Phys. Rev. Lett.* **83**, 436 (1999).
- ²⁶D. Jonathan and M. B. Plenio, *Phys. Rev. Lett.* **83**, 1455 (1999); D. Jonathan and M. B. Plenio, *ibid.* **83**, 3566 (1999); G. Vidal, *Phys. Rev. Lett.* **83**, 1046 (2000).

²⁷M. A. Nielsen and J. Kempe, Phys. Rev. Lett. **86**, 5184 (2001).

²⁸A. Chefles, Phys. Rev. A **65**, 052314 (2002).

²⁹K. R. W. Jones, Phys. Rev. A **50**, 3682 (1994).

³⁰G. M. D'Ariano, P. Lo Presti, and P. Perinotti, "Classical randomness in quantum measurements," e-print: quant-ph/0408115.