Quantum nondemolition measurements for quantum information

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We discuss the characterization and properties of quantum nondemolition (QND) measurements on qubit systems. We introduce figures of merit which can be applied to systems of any Hilbert space dimension, thus providing universal criteria for characterizing QND measurements. The controlled-NOT gate and an optical implementation are examined as examples of QND devices for qubits. We also consider the QND measurement of weak values.

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I. INTRODUCTION

The act of measuring a quantum system to acquire information about it must necessarily disturb the system. Quantum nondemolition (QND) measurements [1] allow for the measurement of an observable of a quantum system without introducing a back-action on this observable due to the measurement itself. QND measurements explore the fundamental limitations of measurement and may prove useful in gravity wave detection [2], telecommunications [3], and quantum control [4].

The traditional domain of experimental QND measurements is continuous-variable (CV) quantum optics [5,6]. CV QND measurements are performed using only Gaussian states (those states of the electromagnetic field with a Gaussian Wigner function), working with quadrature components of the field proportional to number and phase in a linearized regime. In this scenario, ideally, the measurement back-action couples only to the conjugate quadrature to that being measured. Such experiments have been characterized by considering the signal-to-noise transfer and conditional variances between various combinations of the input, output, and measurement output of the device. These are known as "T-V" measures [7].

In contrast, discrete variable quantum optics typically deals with two-level quantum systems such as the polarization states of single photons. Quantum bits or "qubits" can be carried by such systems. Progress in the field of quantum information, in particular in the realization of two-qubit gates, has opened a new domain in which QND measurements can be demonstrated. In this scenario measurement of the computational basis would, ideally, only couple back-action into the diagonal basis. Such QND measurements are critical to many key quantum information protocols, such as error correction [8], and enable new computation models [9]. However, in this new domain of low-dimensional quantum systems with arbitrary superposition states, the standard measures used to characterize CV QND measurements are no longer applicable.

Until recently it was only in the domain of cavity quantum electrodynamics that interactions sufficiently strong as to probe the qubit domain could be achieved with optical fields [10,11]. However, the work of Knill, Laflamme, and Milburn [12] introduced the technique of measurement-induced nonlinearities and led to proposals for nondeterministic realizations of QND measurements for traveling fields [13]. In these schemes the nonlinearity is induced through photon counting measurements made on ancilla modes which have interacted with the system modes via linear optics. A demonstration of a QND measurement on a single photonic qubit was recently made by Pryde *et al.* [14]. In this paper we investigate the character, characterization, an optical implementation, and a fundamental application of QND measurements on qubits.

We begin in the next section by describing the basic features that a QND measurement should display. We then propose quantitative measures by which the quality of any QND measurement can be assessed. We consider qubit systems primarily but also discuss the application of these measures to systems of any dimension. In Sec. III we consider the trade-off between the accuracy of the QND measurement and its inevitable back-action on the conjugate observable to that being measured. In Sec. IV we discuss the example of the controlled-NOT (CNOT) gate and show how it can be used to make generalized QND measurement of arbitrary strength. How such measurements can be implemented in optics is described in Sec. V. In Sec. VI we discuss the domain of weak-valued measurements and propose experiments which would highlight some fundamental peculiarities of quantum mechanics.

II. FIDELITY MEASURES FOR QND MEASUREMENTS

A measurement device takes a quantum system in an input state, described in general by the density matrix $\hat{\rho}$, and via an interaction yields a classical measurement outcome i of some particular observable. The quantum system is left in

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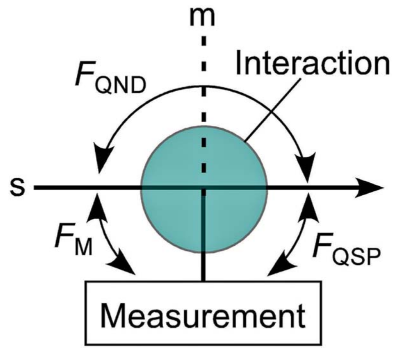

FIG. 1. (Color online) Schematic of a QND measurement. After interaction with the signal s, the meter m carries information about the signal which can be read out by a measurement. The performance against the requirements (i)–(iii) can be assessed by measuring the indicated correlations.

a corresponding output state $\hat{\rho}_i'$. For a direct or destructive measurement of a quantum system it is only required that different eigenvalues of the particular observable of the system be reliably resolved. The output state need not have correlations with either the input state or the measurement results. In contrast, for a QND measurement, as well as giving reliable measurement results, we also require strong correlations in the output state. Conceptually speaking, a QND measurement should satisfy the following criteria (cf. [1]).

(i) *The measurement results should correspond to those expected for the state of the input*. That is, if the input state is an eigenstate of the QND observable, then the measurement outcome corresponding to that eigenstate is achieved with certainty. If the input state is not an eigenstate of the QND observable, then measurement statistics generated by measuring repeated instances of the state should be identical to those that would be obtained via a *direct* measurement of the same observable on the input state.

(ii) *The measurement should not alter the observable being measured*. That is, the Heisenberg evolution through the QND device, *of the Hermitian operator* corresponding to the QND observable, should be the identity for a perfect QND measurement. In the Schödinger picture, for the special case of an eigenstate input, the *state* evolution through the device should be the identity.

(iii) *Repeated measurements should give the same result*. In other words, the QND measurement should be a good quantum-state preparation (QSP) device and ideally should output the eigenstate corresponding to the measurement result.

A schematic representation of a QND measurement illustrating these requirements is shown in Fig. 1. Clearly, these three requirements are not independent. In the following section, we propose quality measures that quantify each of the above requirements. These quality measures apply to arbitrary distributions of input states and serve to quantify realistic QND measurement schemes for which these requirements are not perfectly satisfied.

A. Quantifying performance with fidelities

A QND measurement can be tested relative to the criteria (i)–(iii) by performing repeated measurements of a set of known signal input states $\{\hat{\rho}\}$. Let $\{|\psi_i\rangle, i=1,\ldots,d\}$ be a basis of eigenstates of the measurement of a qudit system with dimension d. There are three relevant probability distributions: p^{in} of the signal input, which consist of the diagonal elements of the signal input density matrix $p_i^{\text{in}}=\langle\psi_i|\hat{\rho}|\psi_i\rangle$ in the basis of eigenstates of the measurement; the distribution p^{out} of the signal output, which consists of the diagonal elements of the signal output density matrix $p_i^{\text{out}}=\langle\psi_i|\hat{\rho}'|\psi_i\rangle$; and the distribution p^{m} of the measurement statistics of the meter. These distributions are all functions of the signal input state. The requirements (i)–(iii) demand *correlations* between these distributions.

To quantify the performance of a QND measurement, we define measures that can be applied to all input states. These measures each compare two probability distributions p and q over the measurement outcomes i, using the (classical) fidelity

$$F(p,q) = \left(\sum_i \sqrt{p_i q_i}\right)^2. \tag{1}$$

Note that $F=1$ for identical distributions, whereas $F=d^{-1}$ for uncorrelated distributions (for example, comparing a lowest entropy distribution $\{1, 0, 0,\ldots, 0\}$ and the highest-entropy distribution $\{d^{-1},d^{-1},\ldots,d^{-1}\}$). For the special case of $d=2$, we also have that $F=0$ for anticorrelated distributions (for example, $\{1, 0\}$ and $\{0, 1\}$). More generally, for qudits, $F=0$ corresponds to orthogonal distributions. As with the probability distributions, these fidelities are also functions of the signal input state.

It is typical for fidelity figures of merit to depend on the input state. To assess the viability of a particular QND device one must thus consider some particular set of input states and evaluate the average or perhaps the minimum fidelity of these states. The particular set of input states that are best to consider will depend on the type of QND device and the purpose for which it is intended.

1. Requirement 1: Measurement fidelity

The first requirement demands that the measurement result be correlated with the state of the input. A device can be tested against this requirement by measuring a set of known states $\{\hat{\rho}\}$ and analyzing the resulting statistics. For example, consider tests involving signal input states that are eigenstates of the observable being measured. Comparing the input distribution p^{in} (which, in this case, are lowest-entropy distributions of the form $\{0, 0,\ldots, 1,\ldots, 0\}$) with the measurement distribution p^{m} (consisting of the probabilities of measuring the result i) quantifies the correlation between the input and the measurement result. However, we note that a QND measurement device is also expected to reproduce the expected measurement statistics for any state, including those that are superpositions of eigenstates, and thus it is necessary to analyze the performance for such non-eigenstate inputs as well.

To quantify the performance of a QND measurement against requirement (i), we define the *measurement fidelity* for the input state $\hat{\rho}$ to be

$$F_{\mathrm{M}} = F(p^{\mathrm{in}}, p^{\mathrm{m}}), \tag{2}$$

which gives the overlap between the signal input and measurement distributions.

As an illustrative example, consider a device where the measurements results are uniformly random and completely uncorrelated with the input states. The measurement statistics are then $p^{\mathrm{m}}=\{d^{-1}, d^{-1}, \ldots, d^{-1}\}$. The resulting measurement fidelity will then range from $d^{-1} \leqslant F_{\mathrm{M}} \leqslant 1$, where the lower bound is obtained for eigenstate inputs and the upper bound is obtained for maximally mixed states or equally weighted superpositions of all eigenstates. Note that, for qubits, a measurement fidelity of 0 is only obtained if the measurement results are completely anticorrelated with the input state.

The measurement fidelity F_{M} can be used to quantify the performance as a measurement device for any signal input state. Of particular interest are the measurement fidelities for the eigenstates, $|\psi_i\rangle$, of the observable, which should ideally give $F_M=1$. It is also important to ensure that all other superposition states produce the correct statistics, and thus the measurement fidelity should be measured for a representative set of states, ideally one which spans the system Hilbert space.

2. Requirement 2: QND fidelity

The second requirement is that the measurement not disturb the observable to be measured—i.e., that the measurement be QND. Note, however, that if the input state is not an eigenstate of the observable being measured, the state must necessarily be altered by the measurement process since the measurement should, ideally, project the signal into an eigenstate. Therefore the signal output of an ideal QND measurement device (when we trace out the meter state) will be in a mixed state where the coherences, $\rho_{ij}=\langle\psi_i|\hat{\rho}|\psi_j\rangle$, $i \neq j$, between different eigenstates have been removed, while leaving the diagonal elements unaltered.

Thus the distribution p^{in} should be preserved by the measurement; i.e., it should be identical to the distribution p^{out}. We compare these two distributions, again using the classical fidelity as a measure, and define the *QND fidelity* for the input state $\hat{\rho}$ to be

$$F_{\mathrm{QND}} = F(p^{\mathrm{in}}, p^{\mathrm{out}}). \tag{3}$$

This measure ranges from 0 to 1, yielding 1 only if the distributions are identical. If, for example, the measurement always produces an eigenstate output but alters ("flips" in the $d=2$ qubit case) the value of i each time, the resulting QND fidelity would be zero.

For input states that are eigenstates of the measurement, the QND fidelity characterizes how well the measurement preserves the measured observable. For input states that are superpositions of eigenstates, the QND fidelity characterizes how well the average populations are preserved.

3. Requirement 3: QSP fidelity

Finally, for a QND measurement we also require that the output state should be the eigenstate $|\psi_i\rangle$ after obtaining the measurement result i. Thus, a measure of quality is needed to characterize how well the output state compares to $|\psi_i\rangle$—i.e., how well the measurement acts as a QSP device.

Let $p^{\mathrm{out}}_{|i\rangle|i}$ denote the *conditional* probability of finding the signal output state to be $|i\rangle$ given that the QND measurement gave the measurement result i. We define the *QSP fidelity*

$$F_{\mathrm{QSP}} = \sum_i p_i^{\mathrm{m}} p^{\mathrm{out}}_{|i\rangle|i}, \tag{4}$$

which is an average fidelity (averaged over all possible measurement outcomes) between the expected and observed conditional probability distributions. This QSP fidelity has the desirable property that it ranges from 0 (if the output state is always orthogonal to the desired outcome) to 1 (if they are equal). If the outcome is uncorrelated with i, the QSP fidelity would have a value of d^{-1}.

For qubits, the QSP fidelity is also known as the *likelihood*, L [15] of measuring the signal to be i given the meter outcome i. We will use this interpretation in the next section to quantify the back-action of a QND measurement.

The most relevant QSP fidelity is obtained when the input is completely unknown—i.e., a completely mixed state. Then this QSP fidelity characterizes how well the measurement device prepares a definite quantum state (labeled by i) given no prior knowledge.

4. Relation between quality measures

As noted above, these three fidelities F_{M}, F_{QND}, and F_{QSP} are interrelated and not independent. Each is used to compare two of the three probability distributions relevant to a QND measurement for a given input state: the distribution p^{in} of input probabilities, the distribution p^{out} of output probabilities, and the distribution p^{m} of measurement outcomes. Any two of these fidelity measures thus sets bounds on the third.

B. Comparing with CV measures

Traditionally, QND measurements have been realized experimentally in quantum optics using Gaussian states (coherent and squeezed states with Gaussian Wigner functions) with a sufficiently large average photon number to allow for a linearized treatment of the quantum noise [16]. Standard quality measures for this particular type of QND measurement have been proposed [7]. In the following, we show how these CV quality measures compare with the general fidelities proposed above (which apply to any d-dimensional system). We note that the available input states in the CV domain do not allow for a complete investigation of the fidelities described above. For example, it is not possible to inject eigenstates of the QND observable because quadrature eigenstates are unphysical.

The CV quality measures are defined in terms of correlations between the input and output states of the quantum system being measured (the signal) and the input and output states of the quantum system used as a meter. Consider a QND observable $\hat{O}$. An example for a qubit system would be the Stokes operator $\hat{S}_1=|H\rangle\langle H|-|V\rangle\langle V|$ and for a CV system the quadrature amplitude fluctuation $\delta\hat{X}=\hat{X}-\langle X\rangle$, where $\hat{X}=\hat{a}+\hat{a}^{\dagger}$ and $\hat{a}$ is the normal boson annihilation operator. Correlation functions are defined as follows. For two systems A and B with QND observables $\hat{O}_A$ and $\hat{O}_B$, respectively, the correlation function between them is defined as

$$C^2(O_AO_B)=\frac{\frac{1}{4}|\langle\hat{O}_A\hat{O}_B\rangle+\langle\hat{O}_B\hat{O}_A\rangle|^2}{\langle\hat{O}_A^2\rangle\langle\hat{O}_B^2\rangle}. \tag{5}$$

Thus, in CV systems, the correlation function between the quadratures of two systems δX_A, δX_B is

$$C^2(\delta X_A\delta X_B)=\frac{\frac{1}{4}|\langle\delta X_A\delta X_B\rangle+\langle\delta X_B\delta X_A\rangle|^2}{\langle\delta X_A^2\rangle\langle\delta X_B^2\rangle}, \tag{6}$$

which provides a measure of the correlation of the two fluctuations, ranging from 0 (uncorrelated) to 1 (perfectly correlated). The quality measures for a CV QND measurement scheme against the criteria listed earlier in this section are the following.

Quality of measurement. The correlation between the input state of the system with fluctuations $\delta X_{\mathrm{s}}^{\mathrm{in}}$ and the output state of the meter with fluctuations δM^{out} is given by the correlation coefficient

$$C_{\mathrm{m}}^2=C^2(\delta X_{\mathrm{s}}^{\mathrm{in}}\delta M^{\mathrm{out}}). \tag{7}$$

This quantity ranges from 0 (for no correlation between the system and the measurement) to 1 (for a perfect measurement); it is thus quantitatively comparable to the measurement fidelity F_{M} when applied to the infinite-dimensional CV system, which then also equals zero for no correlation and unity for perfect correlation. In practice, the correlation C_{m}^2 is not easily measurable in an experiment, and thus the signal-to-noise transfer coefficient T_{M} is typically used [1,7]. T_M is the ratio of the signal to noise of the meter output to the signal to noise of the signal input. This transfer coefficient can be related to C_{m}^2 if one imposes restrictions on the input states—i.e., only Gaussian states and particular choices of squeezing axes (major and minor axes of the ellipse). Given these restrictions there is a direct relationship between F_{M} and T_{M} given by

$$F_{\mathrm{M}}=\sqrt{\frac{2T_{\mathrm{M}}}{1+T_{\mathrm{M}}}}. \tag{8}$$

Quality of QND. The correlation between the input and output states of the system is given by the correlation coefficient

$$C_{\mathrm{s}}^2=C^2(\delta X_{\mathrm{s}}^{\mathrm{in}}\delta X_{\mathrm{s}}^{\mathrm{out}}). \tag{9}$$

This quantity ranges from 0 to 1 and $C_{\mathrm{s}}^2=1$ if the observable X is not disturbed or degraded. The QND fidelity applied to this CV system is quantitatively comparable to C_{s}^2. As for C_{m}^2, C_{s}^2 is not easily measurable in practice, so the ratio of the signal output signal-to-noise to that of the input, the signal transfer coefficient T_{S}, is used. Given the restrictions outlined above there is a direct relationship between F_{QND} and T_{S} given by

$$F_{\mathrm{QND}}=\sqrt{\frac{2T_{\mathrm{S}}}{1+T_{\mathrm{S}}}}. \tag{10}$$

Quality of QSP. The conditional variance $V_{\mathrm{s|m}}$, defined in terms of the fluctuations $\delta X_{\mathrm{s}}^{\mathrm{out}}$ and δM^{out} of the output state of the system and the output state of the meter, respectively, is

$$V_{\mathrm{s|m}}=\langle\delta X_{\mathrm{s}}^{\mathrm{out}^2}\rangle[1-C^2(\delta X_{\mathrm{s}}^{\mathrm{out}}\delta M^{\mathrm{out}})]. \tag{11}$$

This quantity is defined such that $V_{\mathrm{s|m}}=0$ corresponds to perfect correlation between the system and the meter, and $V_{\mathrm{s|m}}<1$ indicates conditional squeezing (quantum correlations). Due to the inclusion of $\langle\delta X_{\mathrm{s}}^{\mathrm{out}^2}\rangle$ in $V_{\mathrm{s|m}}$ as a reference to the shot-noise level, it is not possible to directly compare this quantity to the QSP fidelity. However, if we tried to go the other way and define the QSP fidelity in the CV regime, we find that $\bar{F}_{\mathrm{QSP}}=0$ due to the continuous spectrum of the measurement outcome. To compare directly with CV experiments, the QSP performance for both finite and CV systems could also be quantified by the correlation function between the signal and meter. To this end we note that for qubits $C^2(O_mO_s)=2F_{\mathrm{QSP}}-1$.

III. BACK-ACTION ON THE CONJUGATE VARIABLE

So far we have been assuming that the aim of the QND measurement is to make a nondestructive *projective* measurement of the system. However, such measurements are not the only possibility. We may wish to make a generalized measurement [17] of our system, one that extracts only partial information about the observable in question. In such a situation an additional figure of merit arises: the extent to which the measurement decoheres the observable that is conjugate to the one being measured. For a projective measurement, we expect the conjugate observable to be completely decohered. However, for a partial measurement, the back-action on the conjugate variable need only be sufficient to satisfy complementarity. We can quantify this trade-off for qubits by comparing the distinguishability of the eigenstates of the QND observable via measurements at the meter output with the distinguishability of the eigenstates of the conjugate to the QND observable via measurements at the signal output.

Recall first that the QSP fidelity describes the likelihood L that the meter outcome coincides with the signal outcome. We can thus define the distinguishability of the eigenstates of the QND observable, based on meter measurements, as $K=2L-1$. Now suppose eigenstates of a conjugate observable are injected at the signal input and measurements in the same conjugate basis are made at the signal output. We can define the distinguishability of the conjugate eigenstates based on signal measurements as $\bar{K}=2P_c-1$ where P_c is the probability that the signal out detector correctly identifies the injected eigenstate. Complementarity would suggest that the mutual distinguishability of the two observables should be bounded. This was quantified by Englert [18] who showed that

$$K^2 + \bar{K}^2 \leqslant 1. \tag{12}$$

For an ideal projective QND measurement we would expect $K=1$ and so $\bar{K}=0$. For an ideal generalized QND measurement we might have $K<1$ but would expect the inequality of Eq. (12) to be saturated. We will refer to a QND measurement in which a reduction in K does not result in a corresponding increase in $\bar{K}$ as an *incoherent* QND measurement. An example of an incoherent QND measurement is a "measure and recreate" procedure in which a destructive measurement is made of the system and then a new signal state is generated based on the result of the measurement. Most quantum information applications require coherent QND measurements.

For CV systems back-action on the conjugate variable can be quantified by the generalized uncertainty principle [19] which requires that

$$V_{\mathrm{s|m}} V_{conj} \geqslant 1, \tag{13}$$

where V_{conj} is the variance found in a direct measurement of the conjugate variable on the signal output. For a generalization of this type of an approach to quantifying the measurement-induced back-action see Ref. [20].

IV. PERFORMING QND MEASUREMENTS OF QUBITS WITH A CNOT GATE

We now discuss performing QND measurements in an arbitrary basis (i.e., of an arbitrary observable) and of arbitrary strength on qubits using a controlled-NOT (CNOT) gate. A strong (projective) QND measurement of a qubit in the computational basis can be made with a CNOT gate in the following way.

(i) The target qubit plays the role of the meter; it is prepared in the logical zero state $|\mathbf{0}\rangle$.

(ii) The signal, in an arbitrary state $\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$, is the control qubit.

(iii) The gate is run producing the output state:

$$\alpha|\mathbf{0}\rangle_s|\mathbf{0}\rangle_m + \beta|\mathbf{1}\rangle_s|\mathbf{1}\rangle_m, \tag{14}$$

where s labels the signal (i.e., the control) output and m labels the meter (i.e., the target) output.

(iv) The meter is measured in the computational basis.

With probability $|\alpha|^2$ the meter measurement will give the result "0" and the signal output state will be left in the state $|\mathbf{0}\rangle$. Similarly, with probability $|\beta|^2$ the meter measurement will give the result "1" and the signal state will be left in the state $|\mathbf{1}\rangle$.

The reduced state of both the signal and the meter outputs, if we trace over the other, is

$$\rho_{m/s} = |\alpha|^2|\mathbf{0}\rangle\langle\mathbf{0}| + |\beta|^2|\mathbf{1}\rangle\langle\mathbf{1}|. \tag{15}$$

Thus the probability distributions for measurements in the computational basis will be identical for the input and outputs,

$$p^{in} = p^{out} = p^m = \{|\alpha|^2, |\beta|^2\}, \tag{16}$$

and so against our first two criteria we obtain $F_{\mathrm{M}} = F_{\mathrm{QND}} = 1$, for all input states. The conditional probability that a "0" ("1") result at the meter results in a "0" ("1") result at the signal out is seen from Eq. (14) to be unity. Thus we also have $F_{\mathrm{QSP}}=1$. The CNOT gate then gives an ideal QND measurement.

Similarly the correlation between the signal and meter outputs is

$$C^2(Z_s Z_m) = \frac{|\langle \hat{Z}_s \hat{Z}_m\rangle|^2}{\langle \hat{Z}_s^2\rangle\langle \hat{Z}_m^2\rangle} = 1, \tag{17}$$

where $\hat{Z}_i = |\mathbf{0}\rangle\langle\mathbf{0}|_i - |\mathbf{1}\rangle\langle\mathbf{1}|_i$ is a computational basis measurement on the ith output and the expectation value is taken over the output state [Eq. (14)].

The QND measurement basis can be altered simply by rotating the signal qubit from which ever basis the measurement is desired for, into the computational basis, applying the above protocol, then rotating back to the original basis after.

We now examine altering the strength of the QND measurement. Consider first what occurs if the protocol is followed as above except that we prepare the meter in the diagonal state $(|\mathbf{0}\rangle + |\mathbf{1}\rangle)/\sqrt{2}$. The output state of the gate is then

$$(\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle)_s \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle + |\mathbf{1}\rangle)_m. \tag{18}$$

There is now no correlation between the meter and the signal outputs; tracing over the meter leaves the signal in its original state, and the QND measurement has been "turned off." The probability distributions in the computational basis are now

$$p^{in} = p^{out} = \{|\alpha|^2, |\beta|^2\},$$

$$p^m = \left\{\frac{1}{2}, \frac{1}{2}\right\}, \tag{19}$$

and we find $F_{\mathrm{M}} = 1/2 + \sqrt{|\alpha|^2|\beta|^2}$. For eigenstate inputs we calculate $F_{\mathrm{M}}=1/2$, indicating no correlation. On the other hand, the signal state has not been disturbed and thus $F_{\mathrm{QND}}=1$. The conditional probabilities are

$$p_{0,1}^{out}|_{0,1} = \left\{\frac{1}{2}|\alpha|^2, \frac{1}{2}|\beta|^2\right\}, \tag{20}$$

giving $F_{\mathrm{QSP}}=1/2$ and also $C^2(Z_s Z_m)=0$, both indicating no correlation between the meter and signal output.

More interestingly, let us now see what occurs if the meter is prepared in a state lying in between that producing a strong QND result and that producing no measurement. We thus prepare the meter in the state $\gamma|\mathbf{0}\rangle + \bar{\gamma}|\mathbf{1}\rangle$ where γ is a real number between $1/\sqrt{2}$ and 1 and $\bar{\gamma} = \sqrt{1-\gamma^2}$. The joint meter signal output state is

$$\alpha\gamma|\mathbf{0}\rangle_s|\mathbf{0}\rangle_m + \alpha\bar{\gamma}|\mathbf{0}\rangle_s|\mathbf{1}\rangle_m + \beta\gamma|\mathbf{1}\rangle_s|\mathbf{1}\rangle_m + \beta\bar{\gamma}|\mathbf{1}\rangle_s|\mathbf{0}\rangle_m. \tag{21}$$

The reduced density operators for the signal and meter outputs are

$$\rho_s = (\alpha\gamma|\mathbf{0}\rangle + \beta\bar{\gamma}|\mathbf{1}\rangle)(\alpha^*\gamma\langle\mathbf{0}| + \beta^*\bar{\gamma}\langle\mathbf{1}|) + (\alpha\bar{\gamma}|\mathbf{0}\rangle + \beta\gamma|\mathbf{1}\rangle)(\alpha^*\bar{\gamma}\langle\mathbf{0}| + \beta^*\gamma\langle\mathbf{1}|),$$

$$\rho_m = (\alpha\gamma|\mathbf{0}\rangle + \alpha\bar{\gamma}|\mathbf{1}\rangle)(\alpha^*\gamma\langle\mathbf{0}| + \alpha^*\bar{\gamma}\langle\mathbf{1}|) + (\beta\bar{\gamma}|\mathbf{0}\rangle + \beta\gamma|\mathbf{1}\rangle)(\beta^*\bar{\gamma}\langle\mathbf{0}| + \beta^*\gamma\langle\mathbf{1}|). \quad (22)$$

The relevant probability distributions are now

$$p^{in} = p^{out} = \{|\alpha|^2, |\beta|^2\},$$

$$p^m = \{|\alpha|^2\gamma^2 + |\beta|^2\bar{\gamma}^2, |\beta|^2\gamma^2 + |\alpha|^2\bar{\gamma}^2\}. \quad (23)$$

Now,

$$F_{\mathrm{M}} = [\sqrt{|\alpha|^2(|\alpha|^2\gamma^2 + |\beta|^2\bar{\gamma}^2)} + \sqrt{|\beta|^2(|\beta|^2\gamma^2 + |\alpha|^2\bar{\gamma}^2)}]^2. \quad (24)$$

Eigenstate inputs yield $F_{\mathrm{M}}=\gamma^2$; thus, we can smoothly vary between an ideal QND measurement ($\gamma=1$, $F_{\mathrm{M}}=1$) and no measurment ($\gamma=1/\sqrt{2}$, $F_{\mathrm{M}}=1/2$) by changing γ. The measurement maintains $F_{\mathrm{QND}}=1$ for all γ. The conditional probabilities are

$$p^{out}_{0,1}|_{0,1} = \{|\alpha|^2\gamma^2, |\beta|^2\gamma^2\}, \quad (25)$$

and so $F_{\mathrm{QSP}}=\gamma^2$ and also $C^2(Z_s Z_m)=2\gamma^2-1$. The correlation between the outputs can be smoothly varied between correlated and uncorrelated by tuning the value of γ.

Running in this mode of operation, the CNOT gate is performing a generalized measurement. To see if this is coherent we need to evaluate Eq. (12). We have $K=2\gamma^2-1$. Setting $\alpha=\beta=1/\sqrt{2}$ (a diagonal signal input state) and asking for the probability P_c that ρ_s is measured in a diagonal output state gives the result [from Eq. (22)] $P_c=\gamma\bar{\gamma}+1/2$. Then $\bar{K}=2\gamma\bar{\gamma}$ and $K^2+\bar{K}^2=1$ as required for coherent operation.

V. NONDETERMINISTIC OPTICAL IMPLEMENTATION

We now consider an optical implementation of a qubit QND measurement which is based on linear optical interactions plus a measurement induced nonlinearity. Such an implementation is nondeterministic and, with current technological limitations, relies on coincidence detection. An experimental demonstration of this scheme was presented in Ref. [14]. Here we briefly review the scheme then discuss its characterization in more detail.

A. QND measurement of photon polarization

The aim of our optical QND device is to imprint information about the polarization of a single-signal photon onto the polarization state of a single-meter photon. The polarization degree of freedom of a single photon is a two-dimensional system: a qubit. The net effect of this device is to perform a projective (or generalized) measurement of polarization on a single photon which is then free propagating after this measurement. We describe a scheme for realizing such a measurement nondeterministically using linear optics and single-photon measurement. The optical circuit is shown schematically in Fig. 2. The scheme works on similar principles to those outlined in Sec. IV as its implementation is closely related to the nondeterministic CNOT gate described in [21–23].

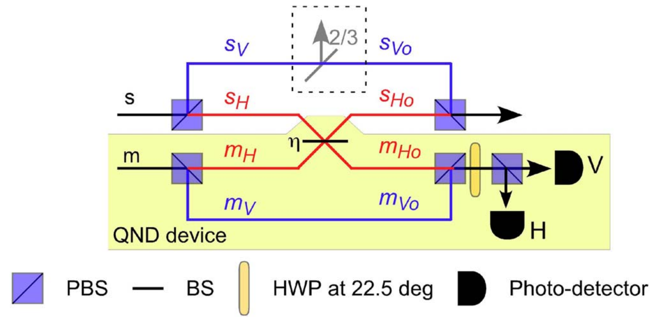

FIG. 2. (Color online) A schematic circuit for the realization of a QND measurement of the polarization of a single photon. Input signal, s, and meter, m, modes containing single-photon states are split into different spatial rails by polarizing beam splitters. The horizontal signal and meter modes are mixed on a beam splitter, leading to nonclassical interference. Detection of a single-meter photon (and absence of a photon at the signal dump port) heralds success of the QND measurement. See text for details.

In the circuit of Fig. 2 a signal and a meter photon are each input from the left. The polarization modes are separated into spatial modes using a polarizing beam splitter (PBS). In the language of optical quantum information the qubit stored in each photon is converted from a polarization encoding into a spatial encoding. The s_H and m_H modes then interfere nonclassically on a beam splitter (BS) with reflectivity η (ηBS). Nonclassical interference between single photons on a BS arises from interference of indistinguishable amplitudes [24]. The four spatial modes are then recombined at a second pair of PBS's as indicated.

When two photons are input into a $\frac{1}{2}$BS the probability of detecting a single photon at each of the outputs is given by the absolute square of the sum of the indistinguishable amplitudes. The two indistinguishable amplitudes correspond to cases where both photons are reflected or both photons are transmitted. Because there is a total π phase shift on reflection, these amplitudes cancel one another and the probability to detect a single photon at each output is zero. This is in contrast to our classical expectation which would lead us to predict that the probability of such an event is $\frac{1}{2}$. For an arbitrary ηBS this same effect means that the probability of detecting a single photon in each output port is simultaneously reduced by the amount

$$\frac{(1-2\eta)^2}{(1-\eta)^2+\eta^2}. \quad (26)$$

In the following discussion we consider a BS where the phase shift on reflection is π on one side and zero on the other without loss of generality.

If we consider just the part of the circuit in Fig. 2 between the two pairs of PBSs, then the Heisenberg equations relating the input modes to the output modes of the circuit are

$$s_{V_O} = s_V, \tag{27}$$

$$s_{H_O} = \sqrt{\eta} s_H + \sqrt{1-\eta} m_H, \tag{28}$$

$$m_{H_O} = \sqrt{1-\eta} s_H - \sqrt{\eta} m_H, \tag{29}$$

$$m_{V_O} = m_V. \tag{30}$$

The goal of the device is that the meter photon interacts with the signal photon in such a way that subsequent measurement of the meter polarization tells us the polarization of the signal. We first consider what the output state of the circuit is for eigenstate signal inputs: $|H\rangle_s$ and $|V\rangle_s$.

We compensate for loss in the meter H component (due to the beam splitter) by adjusting the input state of the meter according to

$$|D(\eta)\rangle_m \equiv \sqrt{\frac{1}{1+\eta}}|H\rangle_m + \sqrt{\frac{\eta}{1+\eta}}|V\rangle_m. \tag{31}$$

This ensures that the meter exits with orthogonal polarizations for different eigenstates. In the case where there is no s_H signal photon, the meter photon exits the meter port with probability $\sqrt{2\eta/(1+\eta)}$. When this happens, the H and V components of the meter state are equal due to the $\sqrt{1-\eta}$ loss in the m_H mode. The goal is then to introduce a π-phase shift in the m_H mode conditional on there being a single photon in the s_H mode so that the meter photon has equal H and V components with a π-phase shift and the meter outputs are orthogonal depending on whether there is a single photon in the s_H mode.

For the signal input $|V\rangle_s$ the total input state (meter and signal) is

$$\begin{aligned}|V\rangle_s|D(\eta)\rangle_m &= |V\rangle_s \otimes \left(\sqrt{\frac{1}{1+\eta}}|H\rangle_m + \sqrt{\frac{\eta}{1+\eta}}|H\rangle_m\right)\\ &= \sqrt{\frac{1}{1+\eta}}|V\rangle_s|V\rangle_m + \sqrt{\frac{\eta}{1+\eta}}|V\rangle_s|H\rangle_m.\end{aligned} \tag{32}$$

We can obtain the output directly using time-reversal symmetry:

$$|\phi_{out}^{V_s}\rangle = \sqrt{\frac{\eta}{1+\eta}}|V\rangle_s(|V\rangle_m + |H\rangle_m) + \sqrt{\frac{1-\eta}{1+\eta}}|H\rangle_s|V\rangle_s. \tag{33}$$

The first term corresponds to a successful operation: one photon exits in each of the signal and meter modes. The second term corresponds to a failure: both photons leave in the signal mode. The probability of success for this signal input P_V is dependent on η,

$$P_V = \sqrt{\frac{2\eta}{1+\eta}}, \tag{34}$$

and $P_V \to 1$ as $\eta \to 1$. When the circuit succeeds, the meter photon is diagonally polarized, $(|D\rangle \equiv |H\rangle + |V\rangle)/\sqrt{2}$, and the signal is vertically polarized. The half-wave plate (HWP) in Fig. 2 is oriented at 22.5° and rotates the meter from $|D\rangle$ to $|V\rangle$, so that the meter and signal then have the same polarization. Measurement of the meter polarization then gives the polarization of the signal without destroying it.

For the other signal eigenstate input $|H\rangle_s$, the total input state is

$$|H\rangle_s|D(\eta)\rangle_m = \sqrt{\frac{1}{1+\eta}}|H\rangle_s|V\rangle_m + \sqrt{\frac{\eta}{1+\eta}}|H\rangle_s|H\rangle_m. \tag{35}$$

We again obtain the output directly using time-reversal symmetry:

$$|\phi_{out}^{H_s}\rangle = \sqrt{\frac{1}{1+\eta}}[(1-2\eta)|H\rangle_s|H\rangle_m - \eta|H\rangle_s|V\rangle_m] + \cdots, \tag{36}$$

where the terms not shown are ones where two photons exit in either the signal or meter mode. The two coefficients are required to be equal (i.e., $1-2\eta=\eta$) so that the meter output state is $(|H\rangle_m - |V\rangle_m)/\sqrt{2} \equiv |A\rangle$. This condition is only satisfied for $\eta=\frac{1}{3}$.

Setting the reflectance $\eta=\frac{1}{3}$, the input state of the meter is required to be

$$|D'\rangle \equiv |D(1/3)\rangle = \frac{\sqrt{3}}{2}|H\rangle_m + \frac{1}{2}|V\rangle_m, \tag{37}$$

and the probability of success for the two eigenstate signal inputs is $P_V=\frac{1}{2}$ and $P_H=\frac{1}{6}$, respectively. For an arbitrary input $\alpha|H\rangle_s+\beta|V\rangle_s$, the probability of success is a weighted average of the two:

$$P_{\alpha|H\rangle_s+\beta|V\rangle_s} = \frac{\alpha^2+3\beta^2}{6}. \tag{38}$$

Success of the circuit is heralded by the detection of a single photon in the meter output—i.e., one photon at one of the two detectors. The success probability is made equal for all input states by introducing the $\frac{2}{3}$ loss (a $\frac{2}{3}$BS), as indicated in Fig. 2; however, an additional detector is then required to monitor this extra output port to check that no photon exits there.

In summary, for a vertical signal state $|V\rangle_s$ the unequal superposition state of the meter [Eq. (31)] combined with the $\frac{2}{3}$ loss experienced by the H component produces $|D\rangle_m$ which is then rotated by a wave plate to $|V\rangle_m$. For a horizontal signal state $|H\rangle_s$, the two-photon nonclassical interference at the $\frac{1}{3}$BS combined with single-photon detection of the meter induces a π phase shift in the H component. This phase shift is transferred to the polarization state of the meter to produce the output $|A\rangle_m$, which is then rotated by the wave plate to $|H\rangle_m$. Finally, note that for the signal in the state $|H\rangle_s$ we make a QND measurement of the photon's presence or absence by measuring the polarization of the meter.

Although closely related to the CNOT gate of Ref. [23] there are a couple of distinct features of the QND gate which we wish to highlight.

(i) Because the target (meter) is in a known state, the loss present in the target arm for the full nondeterministic CNOT gate can be avoided, thus enhancing the success probability of the gate from 1/9 to 1/6 (when signal loss is included).

(ii) The operation of the full CNOT gate is post-selected. However, it is in principle possible to obtain heralded operation from the QND gate by requiring one and only one photon be detected at the meter output (and no photons at the signal loss port, if present).

Because the optical QND gate is based on a CNOT gate generalized measurements can also be implemented. This is correct. As well as performing QND measurements of photon polarization, the device shown in Fig. 2 can also be used to make nondestructive, arbitrary strength measurements of polarization. This is achieved by varying the input state of the meter. Consider what happens when the meter input is $|V\rangle_m$: regardless of the signal polarization, the meter photon travels through the m_V mode and no measurement of the signal is made. A measurement of the signal photon of any strength between no measurement and a projective measurement can be made by preparing the meter in the appropriate real superposition

$$|\psi\rangle_m = a|H\rangle_m + \sqrt{1-a^2}|V\rangle_m, \tag{39}$$

$a \in [0, \sqrt{3}/2]$ (i.e., $|\psi\rangle_m \in \{|D'\rangle_m \rightarrow |V\rangle_m\}$). In this arbitrary strength measurement regime, it is necessary to introduce the additional $\frac{2}{3}$BS into the signal arm to balance the amplitudes in s_{Vo} and s_{Ho}. Because the measurement is no longer projective, unequal amplitudes in these modes would cause the signal state to be rotated, rather than simply cause the success probabilities to be input state dependent.

B. Characterizing nondeterministic, QND coincidence measurements

Our discussion of measures for characterizing QND measurements in Sec. II implicitly assumed that the measurements were deterministic. However, the optical implementation we have been discussing in this section is nondeterministic. How should it be characterized? It seems reasonable that the characterization should be based on only those events for which the device has been predicted to work: when a single meter photon is found after a single signal and meter photon are injected into the device. This question becomes a bit more subtle for the actual experimental situation [14] because these events cannot be unambiguously identified until both the meter *and* the signal photons have been detected—coincidence detection [25,26]. Thus the characterization in [14] was based only on the post-selected events in which photons arrived at both the meter and signal outputs. In particular the probability distributions used to evaluate the various fidelities are obtained in the following ways. For p^{in},

$$p^{in} = (P_H^{in}, P_V^{in}), \tag{40}$$

where $P_i^{in} = R_i^{in}/R_{total}^{in}$ and R_i^{in} is the rate that input photons with polarization i are detected and R_{total}^{in} is the total input photon rate. For p^m,

$$p^m = (P_{H,H}^{out} + P_{H,V}^{out}, P_{V,H}^{out} + P_{V,V}^{out}), \tag{41}$$

where $P_{i,j}^{out} = R_{i,j}^{out}/R_{total}^{out}$ and $R_{i,j}^{out}$ is the rate at which meter photons with polarization i and signal photons with polarization j are simultaneously detected at the output and R_{total}^{out} is the total rate of simultaneously detected photons at the meter and signal outputs. For p^{out},

$$p^{out} = (P_{H,H}^{out} + P_{V,H}^{out}, P_{H,V}^{out} + P_{V,V}^{out}). \tag{42}$$

Finally we can directly write F_{QSP} in terms of coincidence probabilities as

$$F_{QSP} = P_{H,H}^{out} + P_{V,V}^{out}. \tag{43}$$

It has been suggested in Ref. [25] (see also reply [26]) that this post-selection is not justified. Here we argue that it is a sensible application of the figures of merit.

First, the claim of this section is that if two photons are simultaneously incident at the meter and signal inputs of the gate and one and only one photon arrives at the meter output (and no photon departs through the signal loss port if present), then the gate performs a QND measurement of the signal's polarization. However, experimentally it is not possible to reliably post-select successful events based only on the meter output. This is because of technical limitations associated with source and detector efficiency (it is not sure that a photon was present in both the signal and meter inputs) as well as the threshold nature of the detectors (they can only discriminate between zero and more than one photons). By looking at both meter and signal outputs it is possible to reliably post-select only those events corresponding to the theoretical description, as was done in Ref. [14], where the figures of merit confirmed the in principle operation of the gate. Currently, virtually all photonic quantum information demonstrations are of this type. The description of the signal as freely propagating after the QND measurement is valid in the sense that any transformation can be performed on the signal output (including further QND measurements), provided, in the end, only those events where a photon is eventually counted are post-selected.

Second, from an operational point of view, in spite of the limitations of the current experiments, the system can still achieve the goals of a more sophisticated QND device in particular situations. Consider a quantum-key-distribution scheme in which an eavesdropper, Eve, uses the system to make a QND attack on the line. Eve is only ever interested in her meter results in those situations in which (i) she detected a photon and (ii) Bob detected a photon. She knows which events these are from her own records and from analyzing Alice and Bob's public discussion after key exchange. These are precisely the events we suggest are used to calculate the fidelities, and so these are the correct numbers to characterize the effectiveness of her attack in this situation.

We conclude that characterization of a nondeterministic QND gate through coincidence measurements has a clear in-principle and operational interpretation.

VI. MEASUREMENTS OF WEAK VALUES

A. Weak measurements and weak values

As discussed in Sec. IV, a CNOT gate allows a QND measurement to be performed with a variable strength, as measured by the measurement fidelity $F_M = \gamma^2$ for eigenstate inputs. As before γ lies between $1/\sqrt{2}$ and 1 and parametrizes the probe input state $\gamma|0\rangle + \bar{\gamma}|1\rangle$, where $\bar{\gamma} \equiv \sqrt{1-\gamma^2}$. The limit $\gamma = 1$ correspond to a strong, or projective, measurement of the QND observable. Such a measurement collapses the system into an eigenstate. With $1/\sqrt{2} < \gamma < 1$ the measurement is not projective and the disturbance to the system is reduced. The limit $\gamma \to 1/\sqrt{2}$ is that of *weak measurements* where the amount of information obtained is arbitrarily small and the disturbance is also arbitrarily small.

Weak measurements are of fundamental interest because they allow one to measure a *weak value*. A weak value is just the mean value of a weak measurement. That is, it is obtained by averaging over a large ensemble of weak measurement results on identically prepared systems, just as is the mean value of a strong measurement. However, because of the imprecision in each weak measurement result, the size of the ensemble must be correspondingly larger than in the case of strong measurements in order to obtain reliable results.

Simply considering a prepared state $|\psi\rangle$ gives an uninteresting weak value—the same as the strong value for the same quantity:

$$\langle X_{\rm weak}\rangle_\psi = \langle X_{\rm strong}\rangle_\psi = \langle \psi|\hat{X}|\psi\rangle. \tag{44}$$

As realized by Aharonov, Albert, and Vaidman [27], *post-selection* can lead to interesting and nonintuitive weak values that differ from the corresponding strong value. That is, the average is calculated from the subensemble where a *later* strong measurement reveals the state to be $|\phi\rangle$. The post-selected weak value is found to be

$${}_\phi\langle X_{\rm weak}\rangle_\psi = {\rm Re}\frac{\langle \phi|\hat{X}|\psi\rangle}{\langle \phi|\psi\rangle}. \tag{45}$$

This expression is unusual, in that the numerator and denominator are linear in $|\psi\rangle$ and $|\phi\rangle$ rather than bilinear. This has the consequence that the weak value can lie *outside* the range of eigenvalues of $\hat{X}$ [27]. This was soon verified experimentally [28]. However, it is worth remarking that in [28] and later experiments the measurement device used to probe the system was actually another degree of freedom of the system. Thus the experiment can be interpreted within single-particle quantum mechanics (i.e., it displays only semiclassical statistics). In contrast, a weak-value experiment performed using the variable-strength QND measurements we have described in Sec. IV requires two-particle entanglement. Weak values have now been used to analyze a great variety of quantum phenomena [29–37].

A mean value of an observable $\hat{X}$ outside its eigenvalue range cannot occur for a strong measurement of $\hat{X}$, even if the results are post-selected. This constraint can be seen explicitly from the expression for the post-selected strong value,

$${}_\phi\langle X_{\rm strong}\rangle_\psi = \frac{\sum_x |\langle \phi|x\rangle|^2 x |\langle x|\psi\rangle|^2}{\sum_{x'} |\langle \phi|x'\rangle|^2 |\langle x'|\psi\rangle|^2}. \tag{46}$$

The denominator in this expression is the probability $P(\phi|\psi)$ to obtain the final measurement result ϕ irrespective of the result of the intermediate measurement of $\hat{X}$.

From this expression it is also obvious that the average result of the strong measurement of $\hat{X}$, summed over all final measurement results, is the non-post-selected result

$$\sum_\phi {}_\phi\langle X_{\rm strong}\rangle_\psi P(\phi|\psi) = \sum_\phi \sum_x |\langle \phi|x\rangle|^2 x |\langle x|\psi\rangle|^2 = \langle X\rangle_\psi, \tag{47}$$

where $\{|\phi\rangle\}$ is a complete set of orthonormal states. It is less obvious, but also true, that this result holds for weak measurements. In that case, the initial state is hardly disturbed by the measurement of $\hat{X}$ so that $P(\phi|\psi) = |\langle \phi|\psi\rangle|^2$ and

$$\sum_\phi {}_\phi\langle X_{\rm weak}\rangle_\psi P(\phi|\psi) = \sum_\phi {\rm Re}(\langle \psi|\phi\rangle\langle \phi|\hat{X}|\psi\rangle) = \langle X\rangle_\psi. \tag{48}$$

B. Weak values for a qubit

Consider now measuring the logical state of a qubit using a QND device. Because physically this is realized as the photon number (zero or one) of some mode, we will call this observable $\hat{n}$. For simplicity let us consider a single-rail qubit, prepared in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where the value of 0 or 1 in the ket represents both its occupation number and its logical value [38] (as distinct from the dual-rail logic described in Sec. V). Say we were to post-select the weak measurement results on a final (strong) measurement of the qubit in the logical basis, yielding result $m \in \{0,1\}$. Then Eqs. (45) and (46) show that the weak value would again be the same as the strong value and would agree with the final result

$${}_m\langle n_{\rm weak}\rangle_\psi = {}_\psi\langle n_{\rm strong}\rangle_\psi = m. \tag{49}$$

To obtain an interesting weak value, outside the eigenvalue range of {0, 1} it is necessary to make the final measurement in a basis different from that of the weak measurement. This motivates considering a final measurement in a basis conjugate to the logical basis. Without loss of generality we can then consider the basis $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and the weak value conditioned on the result $+$,

$${}_+\langle n_{\rm weak}\rangle_\psi = {\rm Re}\frac{\beta}{\alpha+\beta}. \tag{50}$$

Since we can choose $\alpha \approx -\beta$, Eq. (50) can take any value on the real line.

In an experiment, the weak measurement cannot be arbitrarily weak. Thus there will be corrections to Eq. (50) due to the finiteness of $\gamma - 1/\sqrt{2}$. Consider first the case of non-post-selected measurements. As shown in Sec. IV, the entangled state of the system and meter is

$$(\alpha\gamma|0\rangle_s + \beta\bar{\gamma}|1\rangle_s)|0\rangle_m + (\alpha\bar{\gamma}|0\rangle_s + \beta\gamma|1\rangle_s)|1\rangle_m. \quad (51)$$

Thus the probability of measuring the meter to be in state $k=0$ or 1 is

$$P(k|\psi) = \langle\psi|\hat{E}_k|\psi\rangle, \quad (52)$$

where

$$2\hat{E}_k = \hat{1} - (-1)^k(2\gamma^2 - 1)(2\hat{n} - \hat{1}). \quad (53)$$

Thus the mean value of $\hat{n}$ can be determined from the measurement results via

$$2\langle n\rangle_\psi - 1 = \frac{P(1|\psi) - P(0|\psi)}{2\gamma^2 - 1}. \quad (54)$$

Now with post-selection, the mean photon number given by the weak measurement will be

$$2\,{}_\phi\langle n\rangle_\psi - 1 = \frac{P(1|\phi,\psi) - P(0|\phi,\psi)}{2\gamma^2 - 1}. \quad (55)$$

From Eq. (51), these probabilities are

$$P(0|+,\psi) = \frac{1}{2}|\alpha\gamma + \beta\bar{\gamma}|^2/P(+|\psi), \quad (56)$$

$$P(1|+,\psi) = \frac{1}{2}|\alpha\bar{\gamma} + \beta\gamma|^2/P(+|\psi), \quad (57)$$

where

$$P(+|\psi) = (|\alpha\gamma + \beta\bar{\gamma}|^2 + |\alpha\bar{\gamma} + \beta\gamma|^2)/2 = (1 + 4\gamma\bar{\gamma}\,\mathrm{Re}[\alpha\beta])/2.$$

This gives

$${}_+\langle n\rangle_\psi = \frac{|\beta|^2 + 2\gamma\bar{\gamma}\mathrm{Re}[\alpha\beta^*]}{1 + 4\gamma\bar{\gamma}\mathrm{Re}[\alpha\beta]}. \quad (58)$$

It can be shown that the corresponding result for post-selecting on finding the system in state $|-\rangle$ is such that

$$P(+|\psi)_+\langle n\rangle_\psi + P(-|\psi)_-\langle n\rangle_\psi = |\beta|^2. \quad (59)$$

That is, the formalism makes sense for arbitrary strength measurements (arbitrary γ), not just strong and weak measurements.

In the strong-measurement limit, $\gamma \to 1$, so that $\bar{\gamma} \to 0$, this evaluates to

$${}_+\langle n_{\mathrm{strong}}\rangle_\psi = |\beta|^2, \quad (60)$$

which is as expected because the final measurement is conjugate to the strong measurement of $\hat{n}$, so there are no correlations between the final measurement result and the QND measurement result and the post-selection has no effect.

In the weak-measurement limit with $\gamma \to 1/\sqrt{2}$, then as long as $\alpha - \beta \neq 0$, Eq. (58) evaluates to Eq. (50). In a real experiment γ must be finitely greater than $1/\sqrt{2}$; otherwise, it would take an infinite ensemble size to obtain sufficient data to produce a reliable mean value for the weak measurement. It is therefore of interest to know how strong the measurement can be (how large γ can be) while still yielding an interesting weak value (i.e., a negative weak-valued mean photon number). Assuming for simplicity that α and β are real, with $1/\sqrt{2} < \alpha < 1$ and $-1/\sqrt{2} < \beta < 0$, it is easy to show that Eq. (58) is negative as long as

$$\frac{1}{2} < \gamma^2 < \frac{1}{2}(1 + \sqrt{2\alpha^2 - 1}/\alpha). \quad (61)$$

The closer α is to $1/\sqrt{2}$, the more stringent becomes the weakness requirement on γ. For example, if $\alpha=0.8$ and $\beta=-0.6$, then the above inequality gives $\gamma < 0.911$. If we choose $\gamma=0.8$, then Eq. (58) gives ${}_+\langle n\rangle_\psi = -9/7$.

VII. CONCLUSION

In this paper we have explored QND measurements, focusing particularly on qubits and using examples from optics. We introduced general figures of merit for QND operation based on classical fidelities between the measured distributions of the various inputs and outputs of the device. These can be applied regardless of the Hilbert space dimensions. We discussed bounds on the back-action a QND device produces on the conjugate to the QND observable and defined a coherent QND measurement as one that saturates those bounds. As an abstract example we considered qubit QND measurements carried out using a CNOT gate. We showed that this was an example of an ideal QND device, for both projective measurements and generalized measurements of arbitrary strength. As a physical example we considered a nondeterministic optical realization and discussed its characterization under realistic conditions. Finally we looked at a fundamental application of qubit QND: the measurement of weak values. We predict that weak expectation values lying well outside the eigenvalue range of the qubit observable could be obtained using a CNOT gate.

Note added in proof. Recently we have realized experimental weak values similar to those described in Sec. VI [40].

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