Teleportation via multi-qubit channels

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Abstract

We investigate the problem of teleporting an unknown qubit state to a recipient via a channel of $2\mathcal{L}$ qubits. In this procedure a protocol is employed whereby $\mathcal{L}$ Bell state measurements are made and information based on these measurements is sent via a classical channel to the recipient. Upon receiving this information the recipient determines a local gate which is used to recover the original state. We find that the $2^{2\mathcal{L}}$-dimensional Hilbert space of states available for the channel admits a decomposition into four subspaces. Every state within a given subspace is a perfect channel, and each sequence of Bell measurements projects $2\mathcal{L}$ qubits of the system into one of the four subspaces. As a result, only two bits of classical information need be sent to the recipient for them to determine the gate. We note some connections between these four subspaces and ground states of many-body Hamiltonian systems, and discuss the implications of these results towards understanding entanglement in multi-qubit systems.

1 Introduction

In recent times entanglement has come to be recognised as one of the major distinguishing features between quantum systems and classical systems, where it is now seen as being as fundamental as the uncertainty principle. This point of view has arisen due to the realisation that entanglement is a resource to be exploited in the processing of quantum information [1] through processes such as teleportation [2], dense-coding [3] and quantum cryptography [4]. It has also opened new perspectives in other areas such as condensed matter physics, due the to emerging understanding of the relationship between entanglement and quantum critical phenomena [5–8]. As a consequence there has been an intense level of activity in characterising entanglement and studying its properties.

At the level of bi-partite systems entanglement is well understood and can be quantified [9–13]. From studies of three-qubit systems it was realised [14,15] that different categories of entanglement exist in multi-qubit systems, with the specific example of three-way entanglement shown to be essentially different from bi-partite entanglement through the examples of the Greenberger-Horne-Zeilinger (GHZ) and W states. Now, a clear picture of three-qubit entanglement has emerged with the demonstration of three different types of entanglement existing in the three-qubit case, which are characterised by five generically independent invariants [16, 17]. Though the above results for three-qubit systems can in principle be generalised to arbitrary multi-qubit systems, it is technically challenging to undertake. Despite many studies of specific types of multi-qubit entanglement (e.g. [8, 14, 18–26]), a complete description remains elusive.

Our aim in this work is to investigate entanglement in multi-qubit systems through a study of one of its applications, viz. teleportation. The protocol for the procedure is as follows, with a schematic representation shown in Fig. (1). An unknown qubit state is held by a client (Alice), and is to be teleported to a recipient (Bob). Alice and Bob share a quantum channel which is some state of $2\mathcal{L}$ qubits, so the entire system consists of $2\mathcal{L}+1$ qubits. The channel is distributed in such a way that Alice may access $2\mathcal{L}-1$ qubits of the channel while Bob has access to a single qubit of the channel. Alice is to perform $\mathcal{L}$ Bell state measurements on the $2\mathcal{L}$-qubit subsystem which is comprised of her unknown state and $2\mathcal{L}-1$ qubits of the channel. The consequence of this measurement is that Bob is left with a single qubit which is not entangled with the remainder of the system. From the results of the measurements Alice is to send classical information to Bob. Upon receiving this information, and some knowledge of the channel, Bob determines a local unitary operation called a *correction gate* which he applies to his qubit. Any channel for which this procedure exactly reproduces the client state for Bob (i.e. the teleportation is effected with perfect fidelity) we will call a *perfect channel*. One of the aims of this work is to determine the complete set of perfect channels for this protocol. We mention that this protocol is not *tight* in the sense of [27], and consequently does not belong to the classification of teleportation schemes given therein. On the other hand it does bear similarity to the *quantum repeater* described in [28], with the major difference being that we employ Bell measurements whereas local measurements are used in [28].

It is well known that teleportation can be performed with perfect fidelity across a 2-qubit channel when the channel is one of the four Bell states [2]. This is achieved by making a Bell state measurement and then sending two bits of information to the

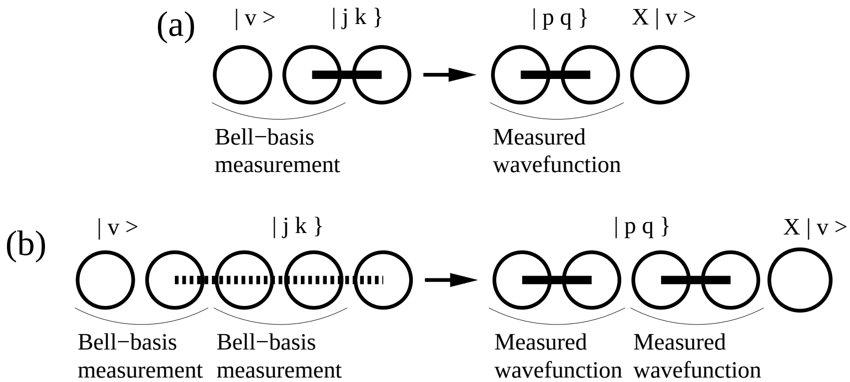

Figure 1: Schematic representation of the teleportation protocol. Circles denote qubit states and two circles joined by a solid line denotes a Bell state. Multi-qubit perfect channels are represented by circles joined by a dashed line. (a) For the case of a 2-qubit channel with client state $|v\rangle$, a Bell measurement is made on the subsystem comprised of the client qubit and the first qubit of the channel. After measurement, the third qubit is left in the state $X\,|v\rangle$, where X is a unitary operator dependent on the measurement outcome. (b) In the case of a 4-qubit channel, two Bell measurements leave the final qubit in the state $X\,|v\rangle$, where X is determined by both measurement outcomes. In general for a $2\mathcal{L}$-qubit channel, a sequence of $\mathcal{L}$ Bell measurements needs to be made to implement the protocol.

recipient via a classical channel, which is then used to determine the correction gate. It is thus clear that teleportation can also be achieved with perfect fidelity using a channel which is a product of $\mathcal{L}$ Bell states, by performing $\mathcal{L}$ successive Bell state measurements. However this is not the most general solution to the problem we have described above, and our analysis below shows some surprising results. The first is that there exist four orthogonal perfect channel subspaces, the direct sum of which is the entire Hilbert space of channels. Also, despite the fact that $\mathcal{L}$ Bell measurements need to be performed by Alice to implement the teleportation, only two bits of classical information need to be sent from Alice to Bob for him to determine the correction gate.

Because perfect channels fall into one of only four subspaces of the channel state space, an interesting question to consider is whether the ground states of common many-body systems fall into these classes. This is indeed the case. For example, our results indicate that all spin singlet states are perfect channels, and so the ground state of the antiferromagnetic Heisenberg model, for a number of different lattices, is a perfect channel, as is the ground state of the one-dimensional Majumdar–Ghosh model [29]. It is also true that the ground state of the model of Affleck, Kennedy, Lieb and Tasaki (AKLT) [30, 31] is a perfect channel, under an equivalent protocol [28, 32, 33]. In identifying the perfect channel states we determine a teleportation-order parameter which provides a measure of the effectiveness of an arbitrary channel. The teleportation-order parameter has close connection with string-order [34], as discussed in [28, 32] in relation to the AKLT model, and is also an example of the string operators discussed in [35]. We mention however that the results of our analysis are independent of the dimension and topology of the lattice on which the qubits are arranged.

We will also show that this analysis extends to formulate a teleportation protocol for the case of 3-qubit channels, and that there is a generalisation for qudits. Finally, we will discuss some implications of these results towards understanding entanglement in multi-qubit systems. The results presented here describe in detail the mathematical aspects which underly the results reported in [36].

2 Teleportation via two-qubit channels

In this section we recall teleportation across a channel of two qubits which exists in one of the four Bell states [2]. While this phenomenon is now well known, the notational conventions we adopt, which are convenient for the following sections, are not standard.

Let $|+\rangle, |-\rangle$ denote the standard basis for a qubit space V such that $|j\rangle$ is an eigenvector of the Pauli matrix σ^z with eigenvalue j. Throughout, we will label ± 1 simply by $\pm$. A natural basis for two coupled qubits is $|j,k\rangle \equiv |j\rangle \otimes |k\rangle$. In making a basis change to the Bell states we define

$$\begin{aligned}
|+:+\} &= \frac{1}{\sqrt{2}}\left(|+,+\rangle + |-,-\rangle\right) \\
|+:-\} &= \frac{1}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle\right) \\
|-:+\} &= \frac{1}{\sqrt{2}}\left(|+,+\rangle - |-,-\rangle\right) \\
|-:-\} &= \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)
\end{aligned}$$

such that we can write

$$|j:k\} = \frac{1}{\sqrt{2}}\left(|+,k\rangle + j\,|-,\overline{k}\rangle\right) \tag{1}$$

where we adopt the notation $\overline{k} = -k$. It is known that each of the Bell basis states are related by a *local* unitary transformation, which we express as

$$|j:k\} = (I \otimes X^{jk}_{pq})\,|p:q\}$$

where

$$\begin{aligned}
&X^{jk}_{jk} = U^0 \\
&X^{+-}_{++} = X^{++}_{+-} = X^{-+}_{--} = X^{--}_{-+} = U^1 \\
&X^{-+}_{++} = X^{++}_{-+} = -X^{+-}_{--} = -X^{--}_{+-} = U^2 \\
&X^{--}_{++} = X^{+-}_{-+} = -X^{++}_{--} = -X^{-+}_{+-} = U^3,
\end{aligned} \tag{2}$$

and the unitary operators U^i are given by

$$\begin{aligned} U^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ U^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ U^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ U^3 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

(Note that the X^{jk}_{pq} could just as easily have been defined in terms of Pauli matrices. We generally prefer to not use Pauli matrix notation, as this eliminates $\sqrt{-1}$ terms which would otherwise appear in many subsequent formulae.) The above is just a statement of the fact that the Bell states are *equivalent*: two states are said to be equivalent if they are equal up to a tensor product of local unitary transformations. Equivalent states have identical entanglement properties. Likewise, we say that two subspaces Y, Z with the same (finite) dimension are equivalent if and only if for a fixed transformation, each $y \in Y$ is equivalent to some $z \in Z$. Moreover two operators are equivalent if they are similar by a transformation which is a tensor product of local unitary transformations.

Define $\nu : \{\pm 1\} \to \mathbb{Z}_2$ by

$$\nu(+1) = 0, \qquad \nu(-1) = 1$$

which satisfies $\nu(ab) = \nu(a) + \nu(b)$. We can express the relations (2) as

$$X^{jk}_{pq} = \delta^{jk}_{pq} (U^1)^{\nu(kq)} (U^2)^{\nu(jp)} \tag{3}$$

where $\delta^{jk}_{pq} = \pm 1$ can be read off from (2). The operators X^{jk}_{pq} satisfy the following properties:

$$\begin{aligned} X^{jk}_{pq} &= \varepsilon^{jk}_{pq} X^{pq}_{jk} = (X^{pq}_{jk})^\dagger && (4) \\ X^{jk}_{pq} X^{pq}_{ab} &= X^{jk}_{ab} && (5) \\ X^{jk}_{pq} X^{ab}_{cd} &= \varepsilon^{jk}_{pq} \varepsilon^{pq}_{jb} X^{jb}_{pq} X^{ak}_{cd} && (6) \\ X^{jk}_{pq} X^{ab}_{cd} &= \delta^{ak}_{jk} \delta^{ab}_{jb} \varepsilon^{jk}_{pq} \varepsilon^{pq}_{ak} X^{ak}_{pq} X^{jb}_{cd} && (7) \\ X^{jk}_{pq} X^{ab}_{cd} &= \delta^{jk}_{pq} \delta^{ab}_{cd} \delta^{(ja)(kb)}_{(pc)(qd)} \varepsilon^{jb}_{pd} X^{(ja)(kb)}_{(pc)(qd)} && (8) \end{aligned}$$

where $\varepsilon^{jk}_{pq} = \varepsilon^{pq}_{jk}$ is defined by

$$\varepsilon^{jk}_{pq} = \begin{cases} -1 & \text{for } j \neq p \text{ and } k \neq q \\ 1 & \text{otherwise} \end{cases} \tag{9}$$

Property (4) is deduced by inspection, while (5) follows from the definition of the X^{jk}_{pq}.

To show (6), first observe that it is true if $k = b$. Assuming $k \neq b$ and using (4,5) we find

$$\begin{aligned} X^{jk}_{pq} X^{ab}_{cd} &= \varepsilon^{jk}_{pq} X^{pq}_{jk} X^{ab}_{cd} \\ &= \varepsilon^{jk}_{pq} X^{pq}_{jb} X^{jb}_{jk} X^{ab}_{ak} X^{ak}_{cd} \\ &= \varepsilon^{jk}_{pq} X^{pq}_{jb} U^1 U^1 X^{ak}_{cd} \\ &= \varepsilon^{jk}_{pq} \varepsilon^{pq}_{jb} X^{jb}_{pq} X^{ak}_{cd}. \end{aligned}$$

Property (7) is proved similarly. To show (8) we calculate

$$\begin{aligned} X^{jk}_{pq} X^{ab}_{cd} &= \delta^{jk}_{pq} \delta^{ab}_{cd} (U^1)^{\nu(kq)} (U^2)^{\nu(jp)} (U^1)^{\nu(bd)} (U^2)^{\nu(ac)} \\ &= \delta^{jk}_{pq} \delta^{ab}_{cd} \varepsilon^{jb}_{pd} (U^1)^{\nu(kqbd)} (U^2)^{\nu(jpac)} \\ &= \delta^{jk}_{pq} \delta^{ab}_{cd} \delta^{(ja)(kb)}_{(pc)(qd)} \varepsilon^{jb}_{pd} X^{(ja)(kb)}_{(pc)(qd)}. \end{aligned}$$

We also find that

$$\begin{aligned} U^1 \otimes U^1 \,|j : k\} &= j \,|j : k\} \\ U^2 \otimes U^2 \,|j : k\} &= k \,|j : k\} \end{aligned}$$

so the eigenvalues of $U^1 \otimes U^1$, $U^2 \otimes U^2$ provide good quantum numbers to label the basis states. (It is easily checked that $U^1 \otimes U^1$ and $U^2 \otimes U^2$ commute.) A measurement which is represented by the action of these two operators is called a Bell measurement, where $(j : k)$ denotes the measurement outcome.

Let $|v\rangle = \alpha\,|+\rangle + \beta\,|-\rangle$ be arbitrary such that

$$|\alpha|^2 + |\beta|^2 = 1$$

and α and β are completely unknown. We call $|v\rangle$ the *client state*. The state which will be used to teleport the client state will be called the *channel*. When the channel is one of the Bell basis states $|j : k\}$ we look to rewrite the total state $|v\rangle \otimes |j : k\}$ as a linear combination of states where the first two qubits are expressed in the Bell basis, i.e.

$$\begin{aligned} &2(I \otimes I \otimes (X^{jk}_{++})^{-1})\,|v\rangle \otimes |j : k\} \\ &\quad = 2\,|v\rangle \otimes |+ : +\} \\ &\quad = \sqrt{2}\,(\alpha\,|+,+,+\rangle + \beta\,|-,+,+\rangle + \alpha\,|+,-,-\rangle + \beta\,|-,-,-\rangle) \\ &\quad = |+ : +\} \otimes (\alpha\,|+\rangle + \beta\,|-\rangle) + |+ : -\} \otimes (\beta\,|+\rangle + \alpha\,|-\rangle) \\ &\qquad\quad + |- : +\} \otimes (\alpha\,|+\rangle - \beta\,|-\rangle) + |- : -\} \otimes (-\beta\,|+\rangle + \alpha\,|-\rangle) \\ &\quad = |+ : +\} \otimes |v\rangle + |+ : -\} \otimes U^1\,|v\rangle + |- : +\} \otimes U^2\,|v\rangle + |- : -\} \otimes U^3\,|v\rangle \\ &\quad = |+ : +\} \otimes X^{++}_{++}\,|v\rangle + |+ : -\} \otimes X^{+-}_{++}\,|v\rangle \\ &\qquad\quad + |- : +\} \otimes X^{-+}_{++}\,|v\rangle + |- : -\} \otimes X^{--}_{++}\,|v\rangle \\ &\quad = |+ : +\} \otimes X^{++}_{++}\,|v\rangle + |+ : -\} \otimes X^{++}_{+-}\,|v\rangle \\ &\qquad\quad + |- : +\} \otimes X^{++}_{-+}\,|v\rangle + |- : -\} \otimes \varepsilon^{--}_{++} X^{++}_{--}\,|v\rangle \end{aligned}$$

and so

$$\begin{aligned} 2\,|v\rangle \otimes |j:k\} &= |+:+\} \otimes X^{jk}_{++}\,|v\rangle + |+:-\} \otimes X^{jk}_{+-}\,|v\rangle \\ &\quad + |-:+\} \otimes X^{jk}_{-+}\,|v\rangle + |-:-\} \otimes \varepsilon^{--}_{++} X^{jk}_{--}\,|v\rangle \\ &= |+:+\} \otimes \varepsilon^{++}_{++} X^{jk}_{++}\,|v\rangle + |+:-\} \otimes \varepsilon^{+-}_{++} X^{jk}_{+-}\,|v\rangle \\ &\quad + |-:+\} \otimes \varepsilon^{-+}_{++} X^{jk}_{-+}\,|v\rangle + |-:-\} \otimes \varepsilon^{--}_{++} X^{jk}_{--}\,|v\rangle\,. \end{aligned}$$

This last expression can be expressed in a compact form:

$$|v\rangle \otimes |j:k\} = \frac{1}{2} \sum_{p,q} |p:q\} \otimes \tilde{X}^{jk}_{pq}\,|v\rangle \tag{10}$$

where $\tilde{X}^{jk}_{pq} = \varepsilon^{pq}_{++} X^{jk}_{pq}$. Thus, when a Bell measurement is made on the first and second qubits by Alice, the system is projected onto a state

$$|p:q\} \otimes \tilde{X}^{jk}_{pq}\,|v\rangle\,.$$

Note that the probabilities for measuring each of the four possible states are equal. The result of the measurement may be communicated to Bob using only two bits of classical information. This, together with knowledge of which channel was used, is sufficient information for Bob to determine the correction gate $\tilde{X}^{pq}_{jk}$, to be implemented in order to recover the client state. Thus the client state has been teleported from Alice to Bob via the channel and classical communication.

3 Teleportation via multi-qubit channels

3.1 Singlet channels: an example of teleportation via multi-qubit channels with perfect fidelity

Our goal is to extend the above construction to the multi-qubit channel case. Here, we will first look at the case when the channel is a $U(2)$ singlet. The Hilbert space for an L-qubit system is given by the tensor product of the local qubit spaces V;

$$V^L \equiv V^{\otimes L} = V_1 \otimes V_2 \otimes \cdots \otimes V_L.$$

Throughout we take L to be even and define $\mathcal{L} = L/2$. Recall that the action of the Lie group $U(2)$ on the space of a single qubit space V is represented by the set of all 2×2 unitary matrices. Given any such matrix $A \in U(2)$, the action extends to the space of L qubits through

$$A \to A^{\otimes L}.$$

A $U(2)$ singlet is any state $|\Psi\rangle \in V^L$ such that for all $A \in U(2)$

$$A^{\otimes L}\,|\Psi\rangle = \exp(i\theta)\,|\Psi\rangle \tag{11}$$

for some real θ. An example of a singlet is given by the Bell state $|-:-\}$.

Let P^{pq} denote the projection onto the Bell state $|p,q\}$. Each projection can be related to the projection onto the $U(2)$ singlet state $|-:-\}$ through

$$P^{pq} = (I \otimes X^{pq}_{--})P^{--}(I \otimes X^{--}_{pq}).$$

Now since $|-:-\}$ is a $U(2)$ singlet then P^{--} is an invariant operator in the sense that

$$(A \otimes A)P^{--}(A^{-1} \otimes A^{-1}) = P^{--} \qquad \forall\, A \in U(2),$$

that is, the action of $U(2)$ commutes with P^{--}. It follows that $P^{--} \otimes I^{\otimes L-2}$ is an invariant operator on the L-fold space V^L. An important result we will use subsequently is Schur's lemma, which asserts that any invariant operator maps an irreducible $U(2)$ invariant space to an isomorphic space by a scalar multiple [37].

Let P^{pq}_r be the projector P^{pq} acting on the rth and $(r+1)$ qubits of the tensor product space and let $(X^{jk}_{pq})_r$ be X^{jk}_{pq} acting on the rth space. Let $|v\rangle$ again be an arbitrary client state, and let the channel $|\Psi\rangle \in V^L$ be an arbitrary singlet state. We denote the space to which the client state belongs by V_0. The initial state of the total system is thus

$$|v^{(0)}\rangle = |v\rangle \otimes |\Psi\rangle\,.$$

Now we employ Schur's lemma, which in particular means that

$$P^{--}_0\, |v\rangle \otimes |\Psi\rangle = \chi\, |-:-\} \otimes |v^{(1)}\rangle$$

for some scalar χ, where $|v^{(1)}\rangle$ is some state in $W^{(1)} = V_2 \otimes V_3 \otimes ... V_L$ which is isomorphic to $|v\rangle$. In other words, if we decompose $W^{(1)}$ into $U(2)$ spaces then $|v^{(1)}\rangle$ belongs to a doublet.

Starting with $|v^{(0)}\rangle$, a Bell measurement is made on $V_0 \otimes V_1$, which is denoted by the projection P^{pq}_0 where $(p:q)$ is the result of the measurement. With reference to the above discussions and notational conventions this means we may write

$$\begin{aligned}
P^{pq}_0|v^{(0)}\rangle &= P^{pq}_0\, |v\rangle \otimes |\Psi\rangle \\
&= (X^{pq}_{--})_1 P^{--}_0 (X^{--}_{pq})_1\, |v\rangle \otimes |\Psi\rangle \\
&= \left(\prod_{r=1}^{L}(X^{pq}_{--})_r\right) P^{--}_0 \left(\prod_{r=1}^{L}(X^{--}_{pq})_r\right) |v\rangle \otimes |\Psi\rangle \\
&= \left(\prod_{r=1}^{L}(X^{pq}_{--})_r\right) P^{--}_0\, |v\rangle \otimes \left(\prod_{r=1}^{L}(X^{--}_{pq})_r\right) |\Psi\rangle \\
&= e^{i\theta} \left(\prod_{r=1}^{L}(X^{pq}_{--})_r\right) P^{--}_0\, |v\rangle \otimes |\Psi\rangle \qquad \text{(since } |\Psi\rangle \text{ is a singlet)} \\
&= e^{i\theta}\chi \left(\prod_{r=1}^{L}(X^{pq}_{--})_r\right) |-:-\} \otimes |v^{(1)}\rangle \quad \text{(by Schur's lemma)} \\
&= e^{i\theta}\chi (X^{pq}_{--})_1\, |-:-\} \otimes \left(\prod_{r=2}^{L}(X^{pq}_{--})_r\right) |v^{(1)}\rangle \\
&= e^{i\theta}\chi\, |p:q\} \otimes \left(\prod_{r=2}^{L}(X^{pq}_{--})_r\right) |v^{(1)}\rangle.
\end{aligned}$$

This procedure can be iterated by taking l consecutive Bell measurements to give

$$P_0^{p_1q_1}P_2^{p_2q_2}....P_{2l-2}^{p_lq_l}|v^{(0)}\rangle = \gamma\,|p_1:q_1\} \otimes |p_2:q_2\} \otimes ... |p_l:q_l\} \otimes \left(\prod_{t=l}^{1}\left(\prod_{r=2l}^{L}(X_{--}^{p_tq_t})_r\right)\right)|v^{(l)}\rangle$$

where γ is a scalar. In particular

$$P_0^{p_1q_1}P_2^{p_2q_2}....P_{L-2}^{p_{\mathcal{L}}q_{\mathcal{L}}}|v^{(0)}\rangle = \gamma\,|p_1:q_1\} \otimes |p_2:q_2\} \otimes ... |p_{\mathcal{L}}:q_{\mathcal{L}}\} \otimes \left(\prod_{t=\mathcal{L}}^{1}(X_{--}^{p_tq_t})_L\right)|v^{(\mathcal{L})}\rangle.$$

Note the notation employed means for any operator Ξ

$$\prod_{j=k}^{1}\Xi_j = \Xi_k....\Xi_2\Xi_1.$$

In each case $|v^{(l)}\rangle \in W^{(l)} = V_{2l} \otimes .... \otimes V_L$ is isomorphic to $|v\rangle$ due to Schur's lemma. However, since $|v^{(\mathcal{L})}\rangle \in W^{(\mathcal{L})} = V_L$ and V_L is an irreducible $U(2)$ space, we must have

$$|v^{(\mathcal{L})}\rangle = |v\rangle_L\,.$$

After the $\mathcal{L}$ Bell basis measurements are made by Alice, Bob needs to apply the correction gate

$$D = \prod_{t=1}^{\mathcal{L}} X_{p_tq_t}^{--}$$

to the Lth qubit in order to recover the client state. In view of (8) we see that

$$D = \kappa X_{pq}^{jk} \tag{12}$$

where $j = k = (-1)^{\mathcal{L}}$, $p = \prod_{t=1}^{\mathcal{L}} p_t$ and $q = \prod_{t=1}^{\mathcal{L}} q_t$. Note that $\kappa = \pm 1$ in (12) is a function of all indices p_t and q_t and can in principle be determined through (8). However its value is inconsequential, as it will only alter the corrected state by a phase. Throughout, whenever such a phase arises we will generically denote it by κ. For ease of notation, we will not explicitly state its dependence on particular indices, although this should be clear from the context.

After Alice has performed the Bell measurements, she need only send two bits of classical information, viz. p and q, to Bob in order for him to determine the correction gate. This is a case where teleportation occurs with perfect fidelity, and shows that *all* singlet states are perfect channels. Next we look to extend this result to cover the most general possibilities.

3.2 A basis of perfect multi-qubit channels

Our first step to classifying the perfect channels is to establish that there exists a basis for the L-qubit Hilbert space V^L in which each basis state is a perfect channel. Since the

Bell states provide a basis for $V \otimes V$ it immediately follows that the set of all vectors of the form

$$| \vec{j} : \vec{k} \} = |j_1 : k_1\} \otimes \cdots \otimes |j_{\mathcal{L}} : k_{\mathcal{L}}\} \tag{13}$$

forms a basis for V^L. Through repeated use of (10) we arrive at

$$|v\rangle \otimes | \vec{j} : \vec{k} \} = \frac{1}{2^{\mathcal{L}}} \sum_{\vec{p},\vec{q}} | \vec{p} : \vec{q} \} \otimes \tilde{X}^{j_{\mathcal{L}} k_L}_{p_{\mathcal{L}} q_{\mathcal{L}}} \dots \tilde{X}^{j_1 k_1}_{p_1 q_1} |v\rangle \tag{14}$$

where the sum is taken over all possible values of $\vec{p}$ and $\vec{q}$. By Alice making pairwise Bell measurements on the first L spaces, a projection is made to a state

$$| \vec{p} : \vec{q} \} \otimes \tilde{X}^{j_{\mathcal{L}} k_{\mathcal{L}}}_{p_{\mathcal{L}} q_{\mathcal{L}}} \dots \tilde{X}^{j_1 k_1}_{p_1 q_1} |v\rangle \,. \tag{15}$$

Given a basis vector $| \vec{j} : \vec{k} \}$ we say that it belongs to the *Bell class* $[j : k]$, j, $k = \pm$ if

$$\prod_{i=1}^{\mathcal{L}} j_i = j, \qquad \prod_{i=1}^{\mathcal{L}} k_i = k.$$

There are four distinct Bell classes. Given an arbitrary vector

$$|\Phi\rangle = \sum_{\vec{j},\vec{k}} \Gamma_{\vec{j},\vec{k}} | \vec{j} : \vec{k} \} \tag{16}$$

we say that $|\Phi\rangle$ belongs to the Bell class $[j : k]$ if, for $\Gamma_{\vec{j}\vec{k}}$ non-zero, then $| \vec{j} : \vec{k} \} \in [j : k]$. In other words, $|\Phi\rangle$ belongs to the Bell class $[j : k]$ if it is a linear combination of basis vectors (13) of Bell class $[j : k]$. It is clear that the notion of Bell classes leads to a vector space decomposition

$$V^L = V^L_{[+:+]} \oplus V^L_{[+:-]} \oplus V^L_{[-:+]} \oplus V^L_{[-:-]}$$

and we refer to each $V^L_{[j:k]}$ as a Bell subspace. In view of (14) we arrive at

$$|v\rangle \otimes |\Phi\rangle = \frac{1}{2^{\mathcal{L}}} \sum_{\vec{p},\vec{q},\vec{j},\vec{k}} \Gamma_{\vec{j}\vec{k}} | \vec{p} : \vec{q} \} \otimes \tilde{X}^{j_{\mathcal{L}} k_L}_{p_{\mathcal{L}} q_{\mathcal{L}}} \dots \tilde{X}^{j_1 k_1}_{p_1 q_1} |v\rangle \tag{17}$$

where the sum is taken over all possible values of $\vec{j}$, $\vec{k}$, $\vec{p}$ and $\vec{q}$. Making pairwise measurements on the first L spaces then projects out a state

$$\mathcal{N} \sum_{\vec{j},\vec{k}} \Gamma_{\vec{j}\vec{k}} | \vec{p} : \vec{q} \} \otimes \tilde{X}^{j_{\mathcal{L}} k_{\mathcal{L}}}_{p_{\mathcal{L}} q_{\mathcal{L}}} \dots \tilde{X}^{j_1 k_1}_{p_1 q_1} |v\rangle \tag{18}$$

where $\mathcal{N}$ is a normalisation factor. Now suppose $|\Phi\rangle \in V^L_{[j:k]}$. Again appealing to (8) and using the fact that $\Gamma_{\vec{j}\vec{k}} = 0$ for $| \vec{j} : \vec{k} \} \notin [j : k]$, then up to a phase we can express (18) as

$$| \vec{p} : \vec{q} \} \otimes \tilde{X}^{jk}_{pq} |v\rangle$$

where $[j:k]$ is the Bell class of the channel, and $[p:q]$ is the Bell class of the measurement (more precisely, the Bell class of the measured tensor product of Bell states). As in the case of singlet channels, Alice again just needs to communicate the Bell class of her measurement (i.e. two bits of classical information) to Bob for him to determine the correction gate. Here we assume, as in the case of teleportation across a single Bell state, that the Bell class of the channel is known to Bob.

A characteristic of the states of Bell subspaces is that they are simultaneous eigenvectors of the operators

$$\Upsilon^1 = \prod_{p=1}^{L} U_p^1, \qquad \Upsilon^2 = \prod_{q=1}^{L} U_q^2,$$

and therefore an eigenstate of the product

$$\Upsilon^3 = \left(\prod_{p=1}^{L} U_p^1\right)\left(\prod_{q=1}^{L} U_q^2\right) = \prod_{p=1}^{L} U_p^3.$$

Note also that

$$[\Upsilon^\alpha,\, \Upsilon^\beta] = 0 \qquad \forall\, \alpha,\, \beta = 1,2,3.$$

It is apparent that each space $V_{[j:k]}^L$ is a stabiliser space for the set of operators $\{j\Upsilon^1,\, k\Upsilon^2\}$. In fact our protocol can be re-expressed as a multi-qubit generalisation of the stabiliser description of teleportation given in [38]. Another result that can be immediately deduced from the above is that any perfect channel $|\varphi\rangle \in V_{[j:k]}^L$ is maximally locally disordered; i.e.,

$$\langle\varphi|\, U_p^\alpha \,|\varphi\rangle = 0 \quad \forall\, \alpha = 1,2,3, \quad \forall\, p = 1,\ldots,L. \tag{19}$$

The result follows from the fact that $|\varphi\rangle$ is an eigenstate of each Υ^α, the Υ^α are self-adjoint, and

$$\Upsilon^\alpha U_p^\beta = -U_p^\beta \Upsilon^\alpha \qquad \text{for} \qquad \alpha \neq \beta. \tag{20}$$

Note that (20) also shows that the Bell subspaces are equivalent.

Let $\mathcal{P}_{mn}$ denote the permutation operator which permutes the mth and nth qubits of the tensor product space V^L. These operators provide a representation of the symmetric group. Since the $\mathcal{P}_{mn}$ commute with the Υ^α, it follows that each of the subspaces $V_{[j:k]}^L$ is invariant under the action of the symmetric group. Thus given any perfect channel, it can be used by Alice and Bob to achieve unit fidelity teleportation independent of which qubit of the channel is Bob's, how Alice chooses to pair the qubits in making the Bell basis measurements, and the order in which she performs the measurements. In particular, it is not necessary that her first measurement involves the client state.

We mention here that unlike the $\mathcal{L} = 1$ case, the probabilities for the measurements that may be made by Alice are not necessarily equal in the case of general $\mathcal{L}$. An illustration of this fact is given by the example in the Appendix. It is true however that the probability a measurement made by Alice is of the Bell class $[j : k]$ is always 1/4, independent of j, k or L. To show this we first construct projection operators $P_{[j:k]}$ onto the subspaces $V^L_{[j:k]}$ by

$$\begin{aligned} P_{[j:k]} &= \frac{1}{4}(I + j\Upsilon^1)(I + k\Upsilon^2) \\ &= \frac{1}{4}\left(I + j\Upsilon^1 + k\Upsilon^2 + jk\Upsilon^3\right). \end{aligned}$$

Given any perfect channel $|\varphi_{[p:q]}\rangle$ of Bell class $[p : q]$ and client state $|v\rangle$ the density matrix is

$$\rho = |v\rangle\langle v| \otimes |\varphi_{[p:q]}\rangle\langle\varphi_{[p:q]}|.$$

The probability $\mathcal{P}_{[j:k]}$ that Alice makes a measurement of Bell class $[j : k]$ is given by

$$\begin{aligned} \mathcal{P}_{[j:k]} &= \mathrm{tr}[(P_{[j:k]} \otimes I)\rho] \\ &= \frac{1}{4}\left(\mathrm{tr}[\rho] + \mathrm{tr}[j(\Upsilon^1 \otimes I)\rho] + \mathrm{tr}[k(\Upsilon^2 \otimes I)\rho] + \mathrm{tr}[jk(\Upsilon^3 \otimes I)\rho]\right) \\ &= \frac{1}{4}\left(1 + \mathrm{tr}[j(\Upsilon^1 \otimes U^1)\rho U^1_L] + \mathrm{tr}[k(\Upsilon^2 \otimes U^2)\rho U^2_L] - \mathrm{tr}[jk(\Upsilon^3 \otimes U^3)\rho U^3_L]\right) \\ &= \frac{1}{4}\left(1 + \mathrm{tr}[j(U^1 \otimes \Upsilon^1)\rho U^1_L] + \mathrm{tr}[k(U^2 \otimes \Upsilon^2)\rho U^2_L] - \mathrm{tr}[jk(U^3 \otimes \Upsilon^3)\rho U^3_L]\right) \\ &= \frac{1}{4}\left(1 + \mathrm{tr}[jpU^1_1\rho U^1_L] + \mathrm{tr}[kqU^2_1\rho U^2_L] - \mathrm{tr}[jkpqU^3_1\rho U^3_L]\right) \\ &= \frac{1}{4}\left(1 + jp\left\langle v|U^1_1|v\right\rangle\langle\varphi_{[p:q]}|U^1_L|\varphi_{[p:q]}\rangle\right. \\ &\qquad \left. + kq\left\langle v|U^2_1|v\right\rangle\langle\varphi_{[p:q]}|U^2_L|\varphi_{[p:q]}\rangle - jkpq\left\langle v|U^3_1|v\right\rangle\langle\varphi_{[p:q]}|U^3_L|\varphi_{[p:q]}\rangle\right) \\ &= \frac{1}{4} \end{aligned}$$

where the last line follows from (19).

It is of interest to consider how the above results relate to common physical models. It was shown earlier that all singlet states are perfect channels, and since they form a subspace (for fixed L), they must belong to the same Bell class. The ground state of the antiferromagnetic Heisenberg model is a singlet, as is the ground state of the one-dimensional Majumdar-Ghosh model [29], so each is a perfect channel. In one-dimension the Heisenberg model is gapless, so any physical realisation would be susceptible to errors arising from thermal fluctuations. One way to reduce errors is to use a gapped system, which is the case for the Heisenberg model on a two-leg ladder lattice [39] as well as the one-dimensional Majumdar-Ghosh model. For the Heisenberg model on the two-dimensional Kagome lattice the system is gapless, but the elementary gapless excitations are also singlets [40]. In this instance error due to thermal fluctuation is again reduced since all singlet states belong to the same Bell class. The existence of such subspaces for which all states provide perfect fidelity teleportation is reminiscent of decoherence free subspaces used to encode logical qubits which are immune to decoherence effects [41].

3.3 Cluster states

Cluster states were introduced in [42] as examples of multi-qubit states with maximal connectedness and high persistency of entanglement. The utilisation of these states for one-way quantum computation has been studied in [43–45]. We will indicate here how each of the one-dimensional cluster states is equivalent to a particular Bell class state.

The one-dimensional cluster states may be defined as the L-qubit states

$$|\phi^{(L)}\rangle = \frac{1}{2^{\mathcal{L}}}\left[\bigotimes_{j=1}^{L-1}\Big(|+\rangle_j + |-\rangle_j U_{j+1}^2\Big)\right]\otimes(|+\rangle_L + |-\rangle_L). \tag{21}$$

Consider the set of operators K_j defined by

$$\begin{aligned} K_1 &= U_1^1 U_2^2, \\ K_j &= U_{j-1}^2 U_j^1 U_{j+1}^2, \qquad j = 2,\ldots,L-1, \\ K_L &= U_{L-1}^2 U_L^1. \end{aligned} \tag{22}$$

It is straightforward to verify these operators satisfy

$$\begin{aligned} [K_j,\, K_l] &= 0 \qquad j,\, l = 1,\ldots,L, \\ K_j\,|\phi^{(L)}\rangle &= |\phi^{(L)}\rangle \qquad j = 1,\ldots,L. \end{aligned}$$

Define the operators G_1 and G_2 by

$$\begin{aligned} G_1 &= \prod_{j=1}^{\mathcal{L}/2}(K_{4j-3}K_{4j}), \\ G_2 &= \prod_{j=1}^{\mathcal{L}/2}(K_{4j-2}K_{4j-1}) \end{aligned}$$

for $\mathcal{L}/2$ even and

$$\begin{aligned} G_1 &= K_{2\mathcal{L}-1}\prod_{j=1}^{(\mathcal{L}-1)/2}(K_{4j-3}K_{4j}), \\ G_2 &= K_{2\mathcal{L}}\prod_{j=1}^{(\mathcal{L}-1)/2}(K_{4j-2}K_{4j-1}) \end{aligned}$$

for when $\mathcal{L}/2$ is odd. The operators G_1 and G_2 necessarily commute and moreover

$$\begin{aligned} G^1|\phi^{(L)}\rangle &= |\phi^{(L)}\rangle, \\ G^2|\phi^{(L)}\rangle &= |\phi^{(L)}\rangle. \end{aligned}$$

It can be shown that G^1, G^2 are equivalent to Υ^1, Υ^2. We will not give a detailed proof, but rather illustrate some examples. When $L = 6$ we have

$$\begin{aligned} G^1 &= U_1^1 U_2^2 U_3^2 U_4^1 U_5^2 U_4^2 U_5^1 U_6^2 \\ &= -U_1^1 U_2^2 U_3^2 U_4^3 U_5^3 U_6^2, \\ G^2 &= U_1^2 U_2^1 U_3^2 U_2^2 U_3^1 U_4^2 U_5^2 U_6^1 \\ &= -U_1^2 U_2^3 U_3^3 U_4^2 U_5^2 U_6^1 \end{aligned}$$

whilst in the case $L = 8$ we have

$$\begin{aligned} G^1 &= U_1^1 U_2^2 U_3^2 U_4^1 U_5^2 U_4^2 U_5^1 U_6^2 U_7^2 U_8^1 \\ &= U_1^1 U_2^2 U_3^2 U_4^3 U_5^3 U_6^2 U_7^2 U_8^1, \\ G^2 &= U_1^2 U_2^1 U_3^2 U_2^2 U_3^1 U_4^2 U_5^2 U_6^1 U_7^2 U_6^2 U_7^1 U_8^2 \\ &= U_1^2 U_2^3 U_3^3 U_4^2 U_5^2 U_6^3 U_7^3 U_8^2. \end{aligned}$$

For these instances the equivalence of G^1, G^2 to Υ^1, Υ^2 can be deduced by inspection. The result holds true not only for all linear cluster states, but can be generalised to cluster states defined on arbitrary d-dimensional lattices as defined in [42]. However, the proof is made tedious by the fact that the definition of the K_j depends on the choice of cluster in each case, so we omit any details.

The fact that each cluster state is equivalent to some perfect channel means that it has exactly the same entanglement properties as that channel. However, teleportation under our protocol using a cluster state will generally fail because the choice of measurement basis is not optimal. Mathematically, this is because cluster states do not belong to the stabiliser space of $\{\Upsilon^1, \Upsilon^2\}$. This serves to remind that while entanglement is necessary to achieve perfect fidelity teleportation, it is just as necessary that the entanglement be *ordered* with respect to a choice of measurement basis. In the next section we will construct a teleportation-order parameter which quantifies this order, and in turn the efficiency of a channel.

4 Teleportation-order

The manner in which we will construct a teleportation-order parameter is motivated by works studying the AKLT model and the role of the string-order parameter [28, 32]. The starting point for this study is the concept of localisable entanglement.

4.1 Localisable entanglement

Localisable entanglement is defined as the maximal possible entanglement that can be localised between two qubits (or more generally qudits), by an optimal choice of measurements on all other qubits of the system. The concept of localisable entanglement we follow is somewhat looser than that of [28] in that we do not impose that the measurements are local, but rather are Bell measurements. Here we will show that each of the basis states $|\,\vec{j} : \vec{k}\,\}$ has maximal localisable entanglement between *any* two qubits with respect to *any* choice of Bell state measurements on all the other qubits of the system. Once we have established this fact, we then show that the same result holds for all states within a Bell subspace.

Below,

$$|\,\vec{j} : \vec{k}\,\}'$$

is defined to be such that

$$|\,\vec{j} : \vec{k}\,\} = |j_1 : k_1\} \otimes |\,\vec{j} : \vec{k}\,\}'\,.$$

Now if $|\,\vec{j}:\vec{k}\,\}$ belongs to the Bell class $[j:k]$ then $|\,\vec{j}:\vec{k}\,\}'$ belongs to the Bell class $[(jj_1):(kk_1)]$. Next we appeal to (1), which permits us to write

$$\begin{aligned}
|\,\vec{j}:\vec{k}\,\} &= |j_1:k_1\}\otimes|\,\vec{j}:\vec{k}\,\}' \\
&= \frac{1}{\sqrt{2}}\left(|+,k_1\rangle + j_1\,|-,\overline{k_1}\rangle\right)\otimes|\,\vec{j}:\vec{k}\,\}' \\
&= \frac{\kappa}{2^{\mathcal{L}-1/2}}\,|+\rangle\otimes\sum_{\vec{p},\vec{q}}|\,\vec{p}:\vec{q}\,\}\otimes X_{pq}^{(jj_1)(kk_1)}\,|k_1\rangle \\
&\qquad + \frac{j_1\kappa}{2^{\mathcal{L}-1/2}}\,|-\rangle\otimes\sum_{\vec{p},\vec{q}}|\,\vec{p}:\vec{q}\,\}\otimes X_{pq}^{(jj_1)(kk_1)}\,|\overline{k_1}\rangle
\end{aligned}$$

Making pairwise measurements on the interior qubits then projects out a state

$$\frac{1}{\sqrt{2}}\left(|+\rangle\otimes|\,\vec{p}:\vec{q}\,\}\otimes X_{pq}^{(jj_1)(kk_1)}\,|k_1\rangle + j_1\,|-\rangle\otimes|\,\vec{p}:\vec{q}\,\}\otimes X_{pq}^{(jj_1)(kk_1)}\,|\overline{k_1}\rangle\right)$$

It is clear that the two end qubits are disentangled from the rest of the system by this process, and together form the state

$$(I\otimes X_{pq}^{(jj_1)(kk_1)})\,|j_1:k_1\}$$

which is one of the Bell states. Using the properties (4,8) we may rewrite this as

$$\begin{aligned}
(I\otimes X_{pq}^{(jj_1)(kk_1)})\,|j_1:k_1\} &= \kappa'(I\otimes X_{pq}^{jk}X_{++}^{j_1k_1})\,|j_1:k_1\} \\
&= \kappa''(I\otimes X_{pq}^{jk}X_{j_1k_1}^{++})\,|j_1:k_1\} \\
&= \kappa''(I\otimes X_{pq}^{jk})\,|+:+\}
\end{aligned}$$

This final expression only depends on the Bell class of the channel and the Bell class of the measurement, so it extends to linear combinations of states from the same Bell class. It then follows that for any state from a fixed Bell class, keeping in mind the the subspace associated with each Bell class is invariant under the symmetric group, any sequence of Bell measurements on $L-2$ qubits will leave the remaining two qubits in a Bell state. Depending on the context, it can be said that each Bell class state has maximal *localisable entanglement* under Bell measurements, or maximal *entanglement length*. These concepts have been discussed in [28, 32] in relation to the spin-1 AKLT model with spin-1/2 boundary sites. A significant feature of this model is that the system is gapped with finite-range spin correlations, yet has maximal entanglement length.

Following the notational conventions of [33], the Hamiltonian for the AKLT model with spin-1/2 boundaries reads

$$H = h_{1,2} + h_{\mathcal{L}+1,\mathcal{L}} + \sum_{j=2}^{\mathcal{L}-1} H_{j,j+1} \tag{23}$$

where

$$\begin{aligned}
h_{j,k} &= \frac{2}{3}\left(I + \vec{s}_j.\vec{S}_k\right) \\
H_{j,k} &= \vec{S}_j.\vec{S}_k + \frac{1}{3}\left(\vec{S}_j.\vec{S}_k\right)^2.
\end{aligned}$$

Above, $\vec{S}$ is the vector spin-1 operator and $\vec{s}$ is the vector spin-1/2 operator. The ground state for the system can be constructed exactly using $2\mathcal{L} - 2$ virtual qubits to represent the $\mathcal{L} - 1$ local spin-1 spaces [32, 33]. Let

$$P^t = P^{++} + P^{-+} + P^{+-} \in \mathrm{End}(V \otimes V)$$

denote the projection onto the triplet space contained in $V \otimes V$, and let P_r^t denote this operator acting on the rth and $(r+1)$th qubits of V^L. The Hilbert space of states for (23) is the image $W \subset V^L$ of the operator

$$\mathbb{P} = P_2^t P_4^t \ldots P_{L-2}^t, \tag{24}$$

and the ground state is given by

$$|AKLT\rangle = \mathbb{P}\,|\vec{p} : \vec{q}\}$$

where $p_j = q_k = -1 \;\; \forall j, k = 1, \ldots, L$; i.e. the ground state is the projection of a product of virtual 2-qubit singlet states into W. The ground state is a perfect channel under a protocol which employs a Bell measurement on the client state and one boundary spin, followed by a sequence of local measurements on the spin-1 sites [28, 32, 33]. In this procedure the client state is teleported to the other boundary site. The fact that this protocol works with perfect fidelity can be understood through the string-order parameter [28, 32].

For any state $|\vartheta\rangle \in W$ the string-order parameter $\mathcal{S}(\vartheta)$ is defined as

$$\mathcal{S}(\vartheta) = 4\,\langle\vartheta|\, s_1^z \otimes [\otimes_{k=2}^{\mathcal{L}} \exp(i\pi S_k^z)] \otimes s_{\mathcal{L}+1}^z\,|\vartheta\rangle \tag{25}$$

which takes values between -1 and 1. It can be checked that for the ground state

$$\mathcal{S}(AKLT) = -1.$$

We may extend the domain of the local operators $\vec{S}_k$ to act on the direct sum of the triplet and singlet spaces, and represent each local spin-$(1 \oplus 0)$ space by the full tensor product $V \otimes V$ of two virtual qubits. It is then found that

$$\exp(i\pi S_k^z) = -U^2 \otimes U^2.$$

Therefore the expectation value (25) is precisely the expectation value of Υ^2 up to a phase factor of $-(-1)^{\mathcal{L}}$. We thus see that the expectation value of Υ^2 restricted to states in W is equivalent to the string-order parameter for the case of the AKLT model.

4.2 Teleportation-order parameter

By analogy with the string-order parameter, for any state $|\Psi\rangle \in V^L$ we define the teleportation-order parameter $\vec{\mathcal{T}} \in \mathbb{R}^3$ to be

$$\vec{\mathcal{T}}(\Psi) = \frac{1}{\sqrt{3}} \sum_{j=1}^{3} \langle\Psi|\, \Upsilon^j\, |\Psi\rangle\, \vec{e}_j$$

where the $\{\vec{e}_j : j = 1, 2, 3\}$ denotes a set of orthonormal vectors for $\mathbb{R}^3$. Given an arbitrary $|\Psi\rangle \in V^L$ we can make the decomposition into a linear combination of representatives from each Bell subspace:

$$|\Psi\rangle = \sum_{j,k} c_{[j:k]} \left|\Psi_{[j:k]}\right\rangle$$

where $\left|\Psi_{[j:k]}\right\rangle \in V^L_{[j:k]}$ is assumed to be normalised so that $\sum_{j,k} |c_{[j:k]}|^2 = 1$. We can then determine that

$$\begin{aligned} c_{[j:k]} \left|\Psi_{[j:k]}\right\rangle &= P_{[j:k]} |\Psi\rangle \\ &= \frac{1}{4} \left(|\Psi\rangle + j\Upsilon^1 |\Psi\rangle + k\Upsilon^2 |\Psi\rangle + jk\Upsilon^3 |\Psi\rangle \right) \end{aligned} \tag{26}$$

which in turn gives

$$|c_{[j:k]}|^2 = \frac{1}{4} \left(1 + \Omega_{[j:k]}\right) \tag{27}$$

where we have defined

$$\Omega_{[j:k]} = j \langle\Psi| \Upsilon^1 |\Psi\rangle + k \langle\Psi| \Upsilon^2 |\Psi\rangle + jk \langle\Psi| \Upsilon^3 |\Psi\rangle . \tag{28}$$

Inverting these relations yields

$$\begin{aligned} \langle\Psi| \Upsilon^1 |\Psi\rangle &= \sum_{j,k} j|c_{[j:k]}|^2 \\ \langle\Psi| \Upsilon^2 |\Psi\rangle &= \sum_{j,k} k|c_{[j:k]}|^2 \\ \langle\Psi| \Upsilon^3 |\Psi\rangle &= \sum_{j,k} jk|c_{[j:k]}|^2. \end{aligned}$$

For any channel we define the *efficiency of teleportation* ∇ through

$$\begin{aligned} \nabla(\Psi) &= |\vec{\mathcal{T}}(\Psi)|^2 \\ &= \frac{1}{3} \sum_{j=1}^{3} \left\langle \Psi|\Upsilon^j|\Psi \right\rangle^2 \\ &= \frac{1}{3} \left(4 \sum_{j,k} |c_{[j:k]}|^4 - 1 \right). \end{aligned}$$

Since

$$(\Upsilon^j)^2 = I^{\otimes L}$$

then for any state $|\Psi\rangle$

$$-1 \leq \langle\Psi| \Upsilon^j |\Psi\rangle \leq 1, \qquad \forall j = 1, \ldots, L \tag{29}$$

and so the efficiency takes values in the range $0 \leq \nabla \leq 1$. When $\nabla = 0$ we see from (27,28) that the state is an equally weighted linear combination of states from each Bell subspace. At the other extreme, when $\nabla = 1$ it indicates that the state belongs to a Bell subspace.

Next we will show that for any product state the efficiency is bounded:

$$0 \leq \nabla \leq 1/3.$$

Let $|v_j\rangle \in V$ be arbitrary and let

$$|w\rangle = |w_1\rangle \otimes |w_2\rangle \otimes \cdots \otimes |w_{\mathcal{L}}\rangle .$$

where

$$|w_j\rangle = \left|v_{(2j-1)}\right\rangle \otimes |v_{2j}\rangle .$$

It is an exercise to show that

$$\left\langle v_j|U^1|v_j\right\rangle^2 + \left\langle v_j|U^2|v_j\right\rangle^2 - \left\langle v_j|U^3|v_j\right\rangle^2 = 1 \qquad \forall\, j = 1, \ldots, L \tag{30}$$

and that

$$\begin{aligned}
1 &\geq \left\langle v_j|U^1|v_j\right\rangle^2 \geq 0,\\
1 &\geq \left\langle v_j|U^2|v_j\right\rangle^2 \geq 0,\\
1 &\geq -\left\langle v_j|U^3|v_j\right\rangle^2 \geq 0.
\end{aligned}$$

We can then deduce that

$$\begin{aligned}
&\left\langle w_j|U^1 \otimes U^1|w_j\right\rangle^2 + \left\langle w_j|U^2 \otimes U^2|w_j\right\rangle^2 + \left\langle w_j|U^3 \otimes U^3|w_j\right\rangle^2\\
&\quad = \left\langle v_{(2j-1)}|U^1|v_{(2j-1)}\right\rangle^2 \left\langle v_{2j}|U^1|v_{2j}\right\rangle^2 + \left\langle v_{(2j-1)}|U^2|v_{(2j-1)}\right\rangle^2 \left\langle v_{2j}|U^2|v_{2j}\right\rangle^2\\
&\qquad\qquad + \left\langle v_{(2j-1)}|U^3|v_{(2j-1)}\right\rangle^2 \left\langle v_{2j}|U^3|v_{2j}\right\rangle^2\\
&\quad \leq \left\langle v_{2j}|U^1|v_{2j}\right\rangle^2 + \left\langle v_{2j}|U^2_{2j}|v_{2j}\right\rangle^2 - \left\langle v_{2j}|U^3|v_{2j}\right\rangle^2 = 1
\end{aligned}$$

and moreover

$$\begin{aligned}
1 &\geq \left\langle w_j|U^1 \otimes U^1|w_j\right\rangle^2 \geq 0,\\
1 &\geq \left\langle w_j|U^2 \otimes U^2|w_j\right\rangle^2 \geq 0,\\
1 &\geq \left\langle w_j|U^3 \otimes U^3|w_j\right\rangle^2 \geq 0.
\end{aligned}$$

Proceeding analogously it follows by an inductive argument that

$$\sum_{j=1}^{3} \left\langle w|\Upsilon^j|w\right\rangle^2 \leq 1$$

and thus ∇ is bounded above by 1/3 for product states. Alternatively, if $\nabla > 1/3$ for a state $|\Psi\rangle$ then it must certainly be entangled, so ∇ provides a generalised notion of an *entanglement witness* [46–48].

4.3 Fidelity

The above analysis identified channels for which teleportation is achieved with perfect fidelity; viz. those channels which lie in a Bell subspace. In practise, there may be some error in the channel which leads to a loss of fidelity. Below we discuss how such a loss of fidelity may be quantified.

Without loss of generality, since the Bell subspaces are equivalent, let us assume that Alice and Bob believe the channel to lie in $V^L_{[+:+]}$. The protocol requires that Alice makes $\mathcal{L}$ Bell measurements on her subsystem which is comprised of the client qubit and $L-1$ qubits of the channel. In the case of perfect channels, we have shown that the protocol is independent of the way in which she pairs the qubits, nor the order in which she makes the measurements. This is also true for the case of non-perfect channels, which can be seen from eq. (17). So we may simplify the problem by assuming that Alice first makes $\mathcal{L}-1$ Bell measurements on qubits which are contained within the channel, and then the final measurement involving one channel qubit and the client qubit. In view of our earlier discussion on localisable entanglement, the first $\mathcal{L}-1$ measurements project each of the component states $\left|\Psi_{[j:k]}\right\rangle$ onto a product of Bell states tensored with a Bell state shared by Alice and Bob. Suppose that the result of Alice's first $\mathcal{L}-1$ Bell measurements is of class $[r:s]$. The Bell class of each of the component Bell states of the total state shared between Alice and Bob after this measurement can be determined from the Bell class of the measurement, as in Sect. 4.1. Thus, after Alice's first $\mathcal{L}-1$ Bell measurements, the channel shared by Alice and Bob is found to be of the form

$$|\Theta\rangle \otimes |\Psi\rangle$$

where $|\Theta\rangle$ is a state of Bell class $[r:s]$, onto which Alice has projected as a result of her measurement, and

$$|\Psi\rangle = \sum_{j,k} C_{[j:k]} \, |j:k\}$$

with

$$C_{[j:k]} = e^{i\theta_{[j:k]}} c_{[rj:sk]}.$$

The above phase factors $\theta_{[j:k]}$ are unknown, because Alice's measurement results do not determine the overall phases of the components of the remaining shared state. Now the problem has been reduced to the investiagation of teleportation across the 2-qubit channel $|\Psi\rangle$ which Alice, as a result of her measurements, believes to be the Bell state $|r:s\}$.

From (10) we may write

$$|v\rangle \otimes |\Psi\rangle = \frac{1}{2} \sum_{p,q} |p:q\} \otimes \tilde{X}_{pq} \, |v\rangle \tag{31}$$

where

$$\tilde{X}_{pq} = \sum_{j,k} C_{[j:k]} \tilde{X}^{jk}_{pq}.$$

Note that $\tilde{X}_{pq}$ is not necessarily a unitary matrix. The probability $\mathcal{P}_{[pr:qs]}$ of Alice's final measurement result being $(p:q)$, thus making her overall measurement of class $[pr:qs]$, is given by

$$\mathcal{P}_{[pr:qs]} = \frac{1}{4}\left|\langle v|\, \tilde{X}^\dagger_{pq}\tilde{X}_{pq}\, |v\rangle\right|.$$

Now suppose that Alice's final measurement result is $(p:q)$, so she communicates to Bob that the total measurement class is $[pr:qs]$. Upon receiving this information, he would apply the correction gate $(\tilde{X}^{++}_{(pr)(qs)})^\dagger$ in attempting to recover the client state. We define the *fidelity* $\mathcal{F}^{++}_{(pr)(qs)}$ of this attempted teleportation as the square of the magnitude of the overlap between the client state and the state Bob has obtained; i.e

$$\begin{aligned}\mathcal{F}^{++}_{(pr)(qs)} &= \frac{\left|\langle v|\, (\tilde{X}^{(pr)(qs)}_{++})^\dagger\tilde{X}_{pq}\, |v\rangle\right|^2}{\left|\langle v|\, \tilde{X}^\dagger_{pq}\tilde{X}_{pq}\, |v\rangle\right|} \\ &= \frac{\left|\langle v|\, (\tilde{X}^{rs}_{pq})^\dagger\tilde{X}_{pq}\, |v\rangle\right|^2}{\left|\langle v|\, \tilde{X}^\dagger_{pq}\tilde{X}_{pq}\, |v\rangle\right|} \qquad (32)\end{aligned}$$

where we have used (4,8).

We return to eq. (31). Alice believes that $|\Psi\rangle$ is the Bell state $|r:s\}$ (up to an overall phase which we hereafter ignore) so we write for real θ

$$|\Psi\rangle = (I\otimes R(\theta,\hat{n}))\,|r:s\}$$

with

$$R(\theta,\hat{n}) = \cos(\theta/2)I - i\sin(\theta/2)(n_1U^1 + n_2U^2 + in_3U^3)$$

so that

$$\tilde{X}_{pq} = R(\theta,\hat{n})\tilde{X}^{rs}_{pq}. \qquad (33)$$

Using (2) we then find

$$|\Psi\rangle = \left(\cos(\theta/2)I - i\sin(\theta/2)\left(n_1X^{r\overline{s}}_{rs} + (-1)^s n_2X^{\overline{r}s}_{rs} + (-1)^s in_3X^{\overline{r}\overline{s}}_{rs}\right)\right)|r:s\}$$

such that we can idenitfy

$$\begin{aligned}C_{[r:s]} &= \cos(\theta/2)\\ C_{[r:\overline{s}]} &= -in_1\sin(\theta/2)\\ C_{[\overline{r}:s]} &= -(-1)^s in_2\sin(\theta/2)\\ C_{[\overline{r}:\overline{s}]} &= (-1)^s n_3\sin(\theta/2).\end{aligned}$$

Normalisation of $|\Psi\rangle$ requires that $\hat{n} = (n_1,\, n_2,\, n_3)$ is a unit complex vector. In the case that $\hat{n}$ is real then $R(\theta,\hat{n})$ is a unitary matrix corresponding to a rotation of the Bloch sphere by an angle θ about an axis determined by $\hat{n}$. However $R(\theta,\hat{n})$ is not unitary for a generic complex unit vector $\hat{n}$. Substituting (33) into (32) gives

$$\mathcal{F}^{++}_{(pr)(qs)} = \frac{\left|\langle v|\,(\tilde{X}^{rs}_{pq})^{\dagger}R(\theta,\hat{n})\tilde{X}^{rs}_{pq}\,|v\rangle\right|^2}{\left|\langle v|\,(\tilde{X}^{rs}_{pq})^{\dagger}R^{\dagger}(\theta,\hat{n})R(\theta,\hat{n})\tilde{X}^{rs}_{pq}\,|v\rangle\right|}.$$

The minimum fidelity is

$$\begin{aligned}\min\left(\mathcal{F}^{++}_{(pr)(qs)}\right) &= \min_{|v\rangle}\frac{\left|\langle v|\,(\tilde{X}^{rs}_{pq})^{\dagger}R(\theta,\hat{n})\tilde{X}^{rs}_{pq}\,|v\rangle\right|^2}{\left|\langle v|\,(\tilde{X}^{rs}_{pq})^{\dagger}R^{\dagger}(\theta,\hat{n})R(\theta,\hat{n})\tilde{X}^{rs}_{pq}\,|v\rangle\right|}\\ &= \min_{|v\rangle}\frac{\left|\langle v|\,R(\theta,\hat{n})\,|v\rangle\right|^2}{\left|\langle v|\,R^{\dagger}(\theta,\hat{n})R(\theta,\hat{n})\,|v\rangle\right|}\\ &\geq 2\cos^2(\theta/2)-1 \end{aligned} \tag{34}$$

where the above inequality holds for all $\hat{n}$. The proof of this result is given in Appendix B. We now have

$$\begin{aligned}\min\left(\mathcal{F}^{++}_{(pr)(qs)}\right) &\geq 2\cos^2(\theta/2)-1\\ &= 2|C_{[r:s]}|^2-1\\ &= 2|c_{[+:+]}|^2-1\\ &= \frac{1}{2}(\Omega_{[+:+]}-1),\end{aligned}$$

so that generally

$$\mathcal{F}^{jk}_{pq} \geq \frac{1}{2}(\Omega_{[j:k]}-1).$$

That is the quantitity $\Omega_{[j:k]}$, which is simply a linear combination of the components of the teleportation-order parameter, provides a lower bound on the fidelity. The results of simulations for four-qubit channels are shown in Fig. 2 (also in [36]).

5 Teleportation via three-qubit channels

In the above, teleportation was only investigated for an even number of channel qubits. It leaves open the problem of devising a teleportation protocol when the number of channel qubits is odd. Here we won't address this in a general context, but we will show that a protocol does exist for teleportation via three-qubit channels.

We begin by defining an orthonormal basis for three-qubit states which generalises the

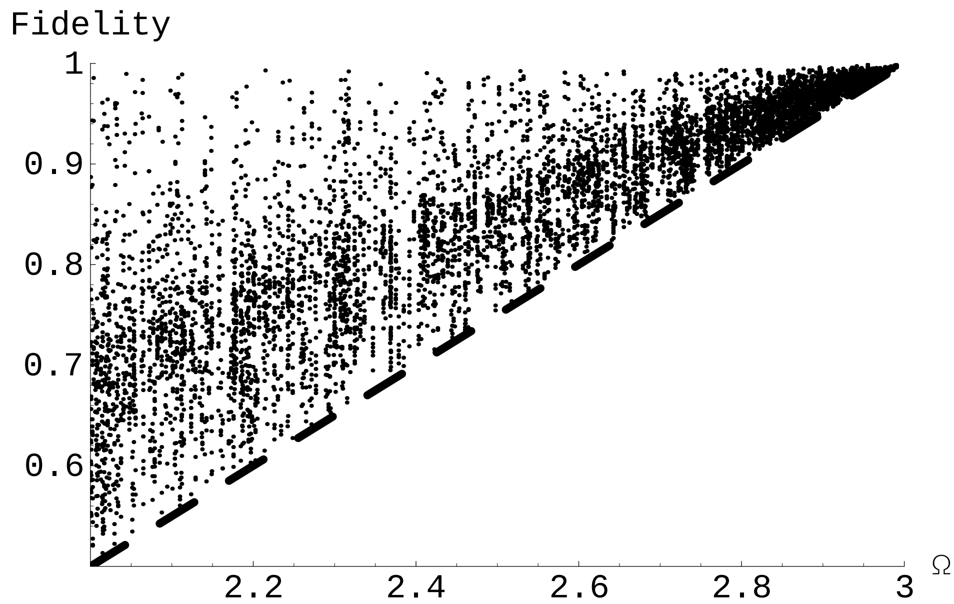

Figure 2: Teleportation fidelity versus the quantity $\Omega_{[j:k]}$ as defined by (28) for all choices of $[j:k]$. The results shown arise from 2000 simulations, with randomly generated client states and 4-qubit channels. The dashed line denotes $\mathcal{F}_{pq}^{jk} = (\Omega_{[j:k]} - 1)/2$, showing there is a lower bound on the fidelity in terms of the components of the teleportation-order parameter, which is independent of the Bell class $[p:q]$ of the measurement outcome.

Bell basis. Let

$$\begin{aligned}
|+:+:+\} &= \frac{1}{\sqrt{2}}\left(|+,+,+\rangle + |-,-,-\rangle\right)\\
|+:+:-\} &= \frac{1}{\sqrt{2}}\left(|+,+,-\rangle + |-,-,+\rangle\right)\\
|+:-:+\} &= \frac{1}{\sqrt{2}}\left(|+,-,+\rangle + |-,+,-\rangle\right)\\
|+:-:-\} &= \frac{1}{\sqrt{2}}\left(|+,-,-\rangle + |-,+,+\rangle\right)\\
|-:+:+\} &= \frac{1}{\sqrt{2}}\left(|+,+,+\rangle - |-,-,-\rangle\right)\\
|-:+:-\} &= \frac{1}{\sqrt{2}}\left(|+,+,-\rangle - |-,-,+\rangle\right)\\
|-:-:+\} &= \frac{1}{\sqrt{2}}\left(|+,-,+\rangle - |-,+,-\rangle\right)\\
|-:-:-\} &= \frac{1}{\sqrt{2}}\left(|+,-,-\rangle - |-,+,+\rangle\right).
\end{aligned}$$

Any basis state can be conveniently expressed as

$$|j:k:l\} = \frac{1}{\sqrt{2}}\left(|+,k,l\rangle + j\,|-,\overline{k},\overline{l}\rangle\right) \tag{35}$$

and satisfies

$$\begin{aligned}\Lambda^1 \left|j:k:l\right\} &= j\left|j:k:l\right\} \\ \Lambda^2 \left|j:k:l\right\} &= k\left|j:k:l\right\} \\ \Lambda^3 \left|j:k:l\right\} &= l\left|j:k:l\right\}\end{aligned}$$

where

$$\begin{aligned}\Lambda^1 &= U^1 \otimes U^1 \otimes U^1 \\ \Lambda^2 &= U^2 \otimes U^2 \otimes I \\ \Lambda^3 &= U^2 \otimes I \otimes U^2.\end{aligned}$$

Above, the Λ^j are mutually commuting self-adjoint operators, so their actions represent a simultaneous measurement. We call such a measurement a three-qubit Bell measurement where $(j:k:l)$ is the measurement outcome. Defining $Y^{jkl}_{pqr} \in \mathrm{End}(V \otimes V)$ by the relation

$$\left|j:k:l\right\} = I \otimes Y^{jkl}_{pqr} \left|p:q:r\right\}$$

we find that

$$Y^{jkl}_{pqr} = Z^k_q \otimes X^{jl}_{pr}$$

where

$$Z^j_j = I, \qquad Z^j_{\bar{j}} = U^1$$

and the X^{jl}_{pr} are as before.

Now consider for an arbitrary client state $|v\rangle = (\alpha|+\rangle + \beta|-\rangle)/\sqrt{2}$

$$\begin{aligned}2|v\rangle \otimes \left|+:+:+\right\} &= \sqrt{2}\,(\alpha|+,+,+,+\rangle + \beta|-,+,+,+\rangle \\ &\qquad +\alpha|+,-,-,-\rangle + \beta|-,-,-,-\rangle) \\ &= \left|+:+:+\right\} \otimes (\alpha|+\rangle + \beta|-\rangle) \\ &\quad + \left|+:-:-\right\} \otimes (\beta|+\rangle + \alpha|-\rangle) \\ &\quad + \left|-:+:+\right\} \otimes (\alpha|+\rangle - \beta|-\rangle) \\ &\quad + \left|-:-:-\right\} \otimes (-\beta|+\rangle + \alpha|-\rangle) \\ &= \left|+:+:+\right\} \otimes |v\rangle + \left|+:-:-\right\} \otimes U^1|v\rangle \\ &\quad - + \left|-:+:+\right\} \otimes U^2|v\rangle + \left|-:-:-\right\} \otimes U^3|v\rangle\,.\end{aligned}$$

By applying the operators U^j to the last qubit of the space we deduce the following relations

$$\begin{aligned}2|v\rangle \otimes \left|+:-:-\right\} &= \left|+:+:-\right\} \otimes U^1|v\rangle + \left|+:-:+\right\} \otimes |v\rangle \\ &\quad + \left|-:+:-\right\} \otimes U^3|v\rangle - \left|-:-:+\right\} \otimes U^2|v\rangle\,,\end{aligned}$$

$$\begin{aligned}2|v\rangle \otimes \left|-:+:+\right\} &= \left|+:+:+\right\} \otimes U^2|v\rangle - \left|+:-:-\right\} \otimes U^3|v\rangle \\ &\quad + \left|-:+:+\right\} \otimes |v\rangle + \left|-:-:-\right\} \otimes U^1|v\rangle\,,\end{aligned}$$

$$
\begin{aligned}
2\,|v\rangle \otimes |-:-:-\} &= -\,|+:+:-\} \otimes U^3\,|v\rangle - |+:-:+\} \otimes U^2\,|v\rangle \\
&\quad + |-:+:-\} \otimes U^1\,|v\rangle + |-:-:+\} \otimes |v\rangle\,.
\end{aligned}
$$

Applying the operator U^1 to the second last qubit of the space in the above four cases gives

$$
\begin{aligned}
2\,|v\rangle \otimes |+:-:+\} &= |+:+:-\} \otimes |v\rangle + |+:-:+\} \otimes U^1\,|v\rangle \\
&\quad + |-:+:-\} \otimes U^2\,|v\rangle + |-:-:+\} \otimes U^3\,|v\rangle\,,
\end{aligned}
$$

$$
\begin{aligned}
2\,|v\rangle \otimes |+:+:-\} &= |+:+:+\} \otimes U^1\,|v\rangle + |+:-:-\} \otimes |v\rangle \\
&\quad + |-:+:+\} \otimes U^3\,|v\rangle - |-:-:-\} \otimes U^2\,|v\rangle\,,
\end{aligned}
$$

$$
\begin{aligned}
2\,|v\rangle \otimes |-:-:+\} &= |+:+:-\} \otimes U^2\,|v\rangle - |+:-:+\} \otimes U^3\,|v\rangle \\
&\quad + |-:+:-\} \otimes |v\rangle + |-:-:+\} \otimes U^1\,|v\rangle\,,
\end{aligned}
$$

$$
\begin{aligned}
2\,|v\rangle \otimes |-:+:-\} &= -\,|+:+:+\} \otimes U^3\,|v\rangle - |+:-:-\} \otimes U^2\,|v\rangle \\
&\quad + |-:+:+\} \otimes U^1\,|v\rangle + |-:-:-\} \otimes |v\rangle\,.
\end{aligned}
$$

The eight relations above can all be expressed in a unified way:

$$
|v\rangle \otimes |j:k:l\} = \frac{1}{2}\sum_{p,q} |p:q:(kq)\} \otimes \tilde{X}^{jl}_{pq}\,|v\rangle\,, \tag{36}
$$

which is a three-qubit channel generalisation of (10). By the same argument as in the two-qubit channel case, we conclude that each $|j:k:l\}$ is a perfect channel for teleportation. After Alice performs a three-qubit Bell measurement which projects the system onto a state

$$
|p:q:(kq)\} \otimes \tilde{X}^{jl}_{pq}\,|v\rangle\,,
$$

two bits of classical information (i.e. p and q) need to be transmitted to Bob for him to determine the correction gate.

Because the teleporation protocol requires that only two bits of classical information be sent to Bob, that part of the Bell measurement represented by Λ_2 becomes redundant. Hence, in analogy with the case where the channel is a state of a system with an even number of qubits, will can still define four Bell classes of channels which give rise to the Hilbert space decomposition

$$
V^3 = V^3_{[+:+]} \oplus V^3_{[+:-]} \oplus V^3_{[-:+]} \oplus V_{[-:-]}
$$

where the class indices $[j:k]$ are the eigenvalues of the operators Λ_1 and Λ_3. Indeed, we can take a generalised form of (36)

$$
|v\rangle \otimes \sum_k \alpha_k\,|j:k:l\} = \frac{1}{2}\sum_{p,q,k} \alpha_k\,|p:q:(kq)\} \otimes \tilde{X}^{jl}_{pq}\,|v\rangle\,, \tag{37}
$$

and perform a reduced three-qubit Bell measurement which is represented by Λ_1 and Λ_3. Above, α_1, α_2 are arbitrary up to the constraint of normalisation. The consequence of the reduced Bell measurement is that it leaves the system in a state of the form

$$\sum_k \alpha_k \left| p : q : (kq) \right\} \otimes \tilde{X}^{jl}_{pq} \left| v \right\rangle ,$$

and once again teleportation can be achieved with perfect fidelity. What this result tells us is that the state of the third qubit of the system (i.e. the second qubit of the channel) is of no consequence in this teleportation protocol. In fact, by a suitable choice of α_1 and α_2 the channel factorises into a tensor product of a Bell state for the first and third qubits of the channel and a disentangled qubit state for the second qubit. Thus this protocol for teleportation via a three-qubit channel is essentially a two-qubit channel protocol as the third qubit can be made redundant.

This raises the question of whether the entanglement of a three-qubit channel can be used to achieve more efficient teleportation than a two-qubit channel. Specifically, can two qubits be teleported via a three-qubit channel? Within the protocol considered here this is not the case. Let $|v\rangle$ and $|w\rangle$ be arbitrary qubit states. Using (36) we calculate

$$\begin{aligned} |v\rangle \otimes |w\rangle \otimes |j : k : l\} &= |v\rangle \otimes \left(\frac{1}{2} \sum_{p,q} |p : q : (kq)\} \otimes \tilde{X}^{jl}_{pq} |w\rangle \right) \\ &= \frac{1}{4} \sum_{p,q,r,s} |r : s : (sq)\} \otimes \tilde{X}^{p(kq)}_{rs} |v\rangle \otimes \tilde{X}^{jl}_{pq} |w\rangle \end{aligned}$$

Now make the change of variable $t = sq$:

$$\begin{aligned} &|v\rangle \otimes |w\rangle \otimes |j : k : l\} \\ &\quad = \frac{1}{4} \sum_{p,r,s,t} |r : s : t\} \otimes \tilde{X}^{p(kst)}_{rs} |v\rangle \otimes \tilde{X}^{jl}_{p(st)} |w\rangle \\ &\quad = \frac{1}{\sqrt{8}} \sum_{r,s,t} |r : s : t\} \otimes \sum_p \frac{1}{\sqrt{2}} \left(\tilde{X}^{p(kst)}_{rs} \otimes \tilde{X}^{jl}_{p(st)} \right) (|v\rangle \otimes |w\rangle) \\ &\quad = \frac{1}{\sqrt{8}} \sum_{r,s,t} |r : s : t\} \otimes \left(\tilde{X}^{+(kst)}_{rs} \otimes \tilde{X}^{jl}_{+(st)} \right) \Theta (|v\rangle \otimes |w\rangle) \end{aligned}$$

where $\Theta = \left(I \otimes I + \kappa \tilde{X}^{-+}_{++} \otimes \tilde{X}^{++}_{-+} \right) / \sqrt{2} = (I \otimes I + \kappa U^2 \otimes U^2) / \sqrt{2}$. Because Θ is not invertible, there is no possibility to effect two-qubit teleportation in this manner.

6 A qudit generalisation

As discussed in the original work [2], it is also possible to teleport qudit states (see also [27]). Let $\{|l\rangle : l = 0, \ldots, d-1\}$ denote a set of orthonormal basis states for a qudit. For ω a fixed primitive dth root of unity, we introduce the permutation and phase matrices defined by

$$\begin{aligned} P\,|l\rangle &= |l+1\rangle\,, \\ Q\,|l\rangle &= \omega^l\,|l\rangle \end{aligned}$$

and set

$$R^{kj} = P^k Q^j.$$

Throughout, the state labels are taken modulo d so for example $|d\rangle \equiv |0\rangle$. A qudit generalisation of Bell states is given by

$$\begin{aligned} |j:k\} &= (I\otimes U^{kj})\,|0:0\} \\ &= \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}\omega^{jl}\,|l\rangle\otimes|l+k\rangle \end{aligned} \tag{38}$$

where

$$|0:0\} = \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}|l\rangle\otimes|l\rangle\,.$$

The generalised Bell states provide a basis which allows us to write

$$|j\rangle\otimes|k\rangle = \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}\omega^{-jl}\,|l:k-j\}\,. \tag{39}$$

Letting $|v\rangle = \sum_{j=0}^{d-1}\alpha_j\,|j\rangle$ denote an arbitrary qudit state we find by using (39)

$$|v\rangle\otimes|0:0\} = \frac{1}{d}\sum_{p,q=0}^{d-1}|p:q\}\otimes R^{q\overline{p}}\,|v\rangle$$

where as before $\overline{p} = -p$. From (38) it follows that

$$|v\rangle\otimes|j:k\} = \frac{1}{d}\sum_{p,q=0}^{d-1}|p:q\}\otimes\tilde{X}^{jk}_{pq}\,|v\rangle \tag{40}$$

with

$$\tilde{X}^{jk}_{pq} = R^{kj}R^{q\overline{p}}.$$

It is apparent that (40) is a qudit generalisation of (10). As the operators R^{jk} generate a group, since $QP = \omega PQ$, so do the $\tilde{X}^{jk}_{pq}$. It follows that our analysis for qubit systems generalises to qudit systems, with the main finding being that for the L-qudit case there are d^2 Bell subspaces of perfect channels. The Bell subspaces are equivalent, with each having dimension d^{L-2}.

7 Conclusion

To conclude, we discuss some aspects of our results in the context of 4-qubit channels. The 16-dimensional Hilbert space V^4 decomposes into four Bell subspaces $V^4_{[j:k]}$, each of dimension four. Since these subspaces are all equivalent, we can focus on the space $V^4_{[+:+]}$. This space is precisely the space G_{abcd} of [49], the generic equivalence class of 4-qubit states representing the orbits arising from stochastic local operations and classical communication (see also [24]). It contains the 4-qubit case of the celebrated Greenberger-Horne-Zeilinger states, for which it has been argued are the only states exhibiting *essential* multi-partite entanglement [19]. It also contains the state of Higuchi and Sudbery [50], which has the largest known average 2-qubit bi-partite entanglement in a 4-qubit system (see also [26]). This state, together with its complex conjugate state, provides a basis for the space of singlets contained in V^4. Further, there are three states in $V^4_{[+:+]}$ which are equivalent to the three 4-qubit cluster states, known to have maximal connectedness and high persistency of entanglement [42]. All the above mentioned states, by representing different forms of multi-partite entanglement, are not equivalent in the sense that they are not related by local unitary transformations. They are however all entirely equivalent for the purpose of teleportation under our prescribed protocol, since they all belong to $V^4_{[+:+]}$. This highlights the fact that the entanglement needed to implement this protocol is of a specific type, which depends on each qubit being maximally entangled with the rest of the system (maximal local disorder). Other forms of entanglement the channel might possess are irrelevant. We do emphasise though that for the channel to be effective, this entanglement has to be ordered with respect to the prescribed measurement basis, which is quantified by the teleportation-order parameter.

Lastly we mention that, besides the one described here, there are many possible teleportation protocols which generalise the original work of [2]; e.g., see [51–53]. It would be useful in future work to identify a correspondence between any given teleportation protocol, and a teleportation-order parameter which signifies when a channel can be used to implement the protocol and effect teleportation with full fidelity. Furthermore, the possibilities for performing teleportation without a shared reference frame, following the ideas developed in [54], also warrant investigation.

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Appendix A

Here we show by example that not all measurement outcomes are equally likely. Let the channel be

$$|\Psi\rangle = \cos(\phi)\,|+:-\}\otimes|-:+\} + \sin(\phi)\,|-:+\}\otimes|+:-\}$$

which is of Bell class $[-:-]$. Using (6,7,17) we may write

$$
\begin{aligned}
4\,|v\rangle \otimes |\Psi\rangle &= \cos(\phi) \sum_{p_1,p_2,q_1,q_2} |\, p_1p_2 : q_1q_2 \,\} \otimes \tilde{X}^{-+}_{p_2q_2}\tilde{X}^{+-}_{p_1q_1}\,|v\rangle \\
&\quad + \sin(\phi) \sum_{p_1,p_2,q_1,q_2} |\, p_1p_2 : q_1q_2 \,\} \otimes \tilde{X}^{+-}_{p_2q_2}\tilde{X}^{-+}_{p_1q_1}\,|v\rangle \\
&= -\cos(\phi) \sum_{p_1,p_2,q_1,q_2} \varepsilon^{p_1q_1}_{++}\varepsilon^{p_2q_2}_{-+} |\, p_1p_2 : q_1q_2 \,\} \otimes X^{++}_{p_2q_2}X^{--}_{p_1q_1}\,|v\rangle \\
&\quad + \sin(\phi) \sum_{p_1,p_2,q_1,q_2} \varepsilon^{p_1q_1}_{++}\varepsilon^{p_2q_2}_{+-} |\, p_1p_2 : q_1q_2 \,\} \otimes X^{++}_{p_2q_2}X^{--}_{p_1q_1}\,|v\rangle \\
&= -(\cos(\phi) - \sin(\phi))\,|{+}{+} : {+}{+}\} \otimes U^3\,|v\rangle \\
&\quad +(\cos(\phi) + \sin(\phi))\,|{+}{+} : {+}{-}\} \otimes U^2\,|v\rangle \\
&\quad +(\cos(\phi) - \sin(\phi))\,|{+}{+} : {-}{+}\} \otimes U^2\,|v\rangle \\
&\quad -(\cos(\phi) + \sin(\phi))\,|{+}{+} : {-}{-}\} \otimes U^3\,|v\rangle \\
&\quad +(\cos(\phi) + \sin(\phi))\,|{+}{-} : {+}{+}\} \otimes U^1\,|v\rangle \\
&\quad -(\cos(\phi) - \sin(\phi))\,|{+}{-} : {+}{-}\} \otimes U^0\,|v\rangle \\
&\quad +(\cos(\phi) + \sin(\phi))\,|{+}{-} : {-}{+}\} \otimes U^0\,|v\rangle \\
&\quad -(\cos(\phi) - \sin(\phi))\,|{+}{-} : {-}{-}\} \otimes U^1\,|v\rangle \\
&\quad -(\cos(\phi) - \sin(\phi))\,|{-}{+} : {+}{+}\} \otimes U^1\,|v\rangle \\
&\quad +(\cos(\phi) + \sin(\phi))\,|{-}{+} : {+}{-}\} \otimes U^0\,|v\rangle \\
&\quad +(\cos(\phi) - \sin(\phi))\,|{-}{+} : {-}{+}\} \otimes U^0\,|v\rangle \\
&\quad -(\cos(\phi) + \sin(\phi))\,|{-}{+} : {-}{-}\} \otimes U^1\,|v\rangle \\
&\quad +(\cos(\phi) + \sin(\phi))\,|{-}{-} : {+}{+}\} \otimes U^3\,|v\rangle \\
&\quad -(\cos(\phi) - \sin(\phi))\,|{-}{-} : {+}{-}\} \otimes U^2\,|v\rangle \\
&\quad +(\cos(\phi) + \sin(\phi))\,|{-}{-} : {-}{+}\} \otimes U^2\,|v\rangle \\
&\quad -(\cos(\phi) - \sin(\phi))\,|{-}{-} : {-}{-}\} \otimes U^3\,|v\rangle\,.
\end{aligned}
$$

From the above it can be seen that for any measurement projecting onto a state

$$|p_1p_2 : q_1q_2\} \otimes U^j\,|v\rangle$$

the probability is either

$$\frac{1}{16}\,(1 + \sin(2\phi))$$

or

$$\frac{1}{16}\,(1 - \sin(2\phi))$$

so not all measurement outcomes are equally likely. It is also easily checked for this case that the probability a measurement is of the Bell class $[j:k]$ is $1/4$, independent of j and k, which is consistent with our earlier result.

Appendix B

Here we prove the inequality (34), viz.

$$\min_{|v\rangle} \frac{|\langle v|\, R(\theta,\hat{n})\, |v\rangle|^2}{|\langle v|\, R^\dagger(\theta,\hat{n}) R(\theta,\hat{n})\, |v\rangle|} \quad \geq \quad 2\cos^2(\theta/2) - 1.$$

Suppose that $|w\rangle$ is a vector for which the miminum is achieved. We can then perform a unitary transformation such that $|w\rangle$ is transformed into the state $|+\rangle$. Under such a unitary transformation, the operator $R(\theta,\hat{n})$ is transformed into $R(\theta,\hat{m})$ for some unit complex vector $\hat{m}$. Importantly, the variable θ is the same for both operators. Setting

$$\Delta(\theta,\hat{m}) = \frac{|\langle +|\, R(\theta,\hat{m})\, |+\rangle|^2}{|\langle +|\, R^\dagger(\theta,\hat{m}) R(\theta,\hat{m})\, |+\rangle|}$$

we now need to show that, for all $\hat{m}$,

$$\Delta(\theta,\hat{m}) \quad \geq \quad 2\cos^2(\theta/2) - 1.$$

We may express a generic $R(\theta,\hat{m})$ as

$$R(\theta,\hat{m}) = \cos(\theta/2) I + \sin(\theta/2) \begin{pmatrix} a & c \\ b & -a \end{pmatrix}$$

where a, b, c are complex parameters subject to the normalisation constraint

$$|a|^2 + \frac{1}{2}(|b|^2 + |c|^2) = 1.$$

Without loss of generality we may impose $0 \leq \theta < \pi$. In terms of these parameters we have

$$\Delta(\theta,\hat{m}) = \frac{|\cos(\theta/2) + a\sin(\theta/2)|^2}{|\cos(\theta/2) + a\sin(\theta/2)|^2 + |b\sin(\theta/2)|^2}. \tag{41}$$

For any fixed a, (41) is minimised by maximising $|b|^2$. We thus choose $c = 0$ leading to

$$\begin{aligned} \Delta(\theta,\hat{m}) &= \frac{|\cos(\theta/2) + a\sin(\theta/2)|^2}{|\cos(\theta/2) + a\sin(\theta/2)|^2 + 2(1-|a|^2)\sin^2(\theta/2)} \\ &= \frac{\cos^2(\theta/2) + |a|^2\sin^2(\theta/2) + 2\Re(a)\sin(\theta/2)\cos(\theta/2)}{1 + (1-|a|^2)\sin^2(\theta/2) + 2\Re(a)\sin(\theta/2)\cos(\theta/2)} \end{aligned} \tag{42}$$

where $\Re(a)$ denotes the real part of a. The above expression is minimised when a is real and given by

$$a = \frac{\sqrt{1 - 4\cos^2(\theta/2)\sin^2(\theta/2)} - 1}{2\sin(\theta/2)\cos(\theta/2)} = \frac{\cos(\theta) - 1}{\sin(\theta)}. \tag{43}$$

Finally, substituting (43) into (42) gives

$$\min_{\hat{m}} \left(\Delta(\theta,\hat{m})\right) = \cos(\theta) = 2\cos^2(\theta/2) - 1$$

as required.

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